AVERAGE AVALANCHE SHAPE AS A PROBE OF NON-EQUILIBRIUM SYSTEMS: HOPES AND PITFALLS

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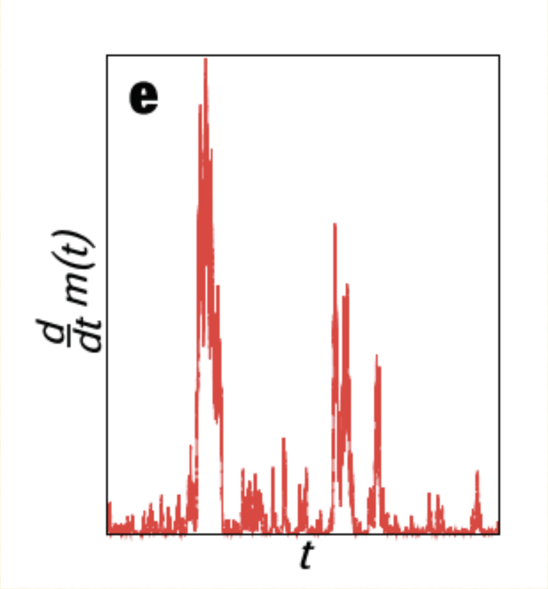
OUTLINE

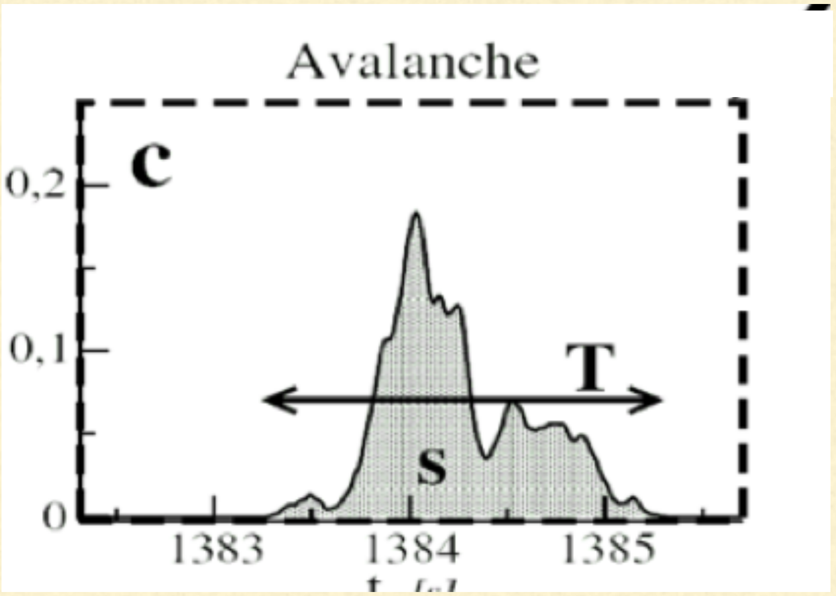
1. intro: avalanches and bridges
2. average avalanche shape: experimental portfolio
3. ABBM model: exact solution
4. ABBM average shape: avalanche and bridge universality
5. asymmetric avalanche shapes: memory and time reverse
6. detailed balance and asymmetry: brownian gyrator
7. non markovian process and incomplete knowledge
8. hopes and pitfalls of linear systems

INTRO

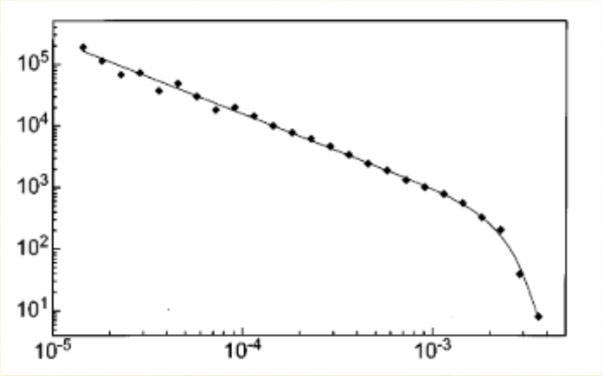
AVALANCHES AND BRIDGES

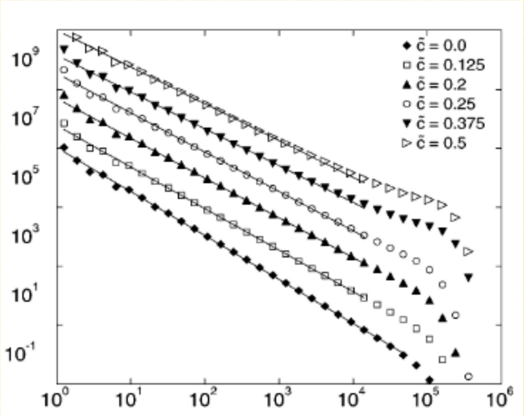
"CRACKLING NOISE" AND AVALANCHE SHAPE





Universal critical exponents?





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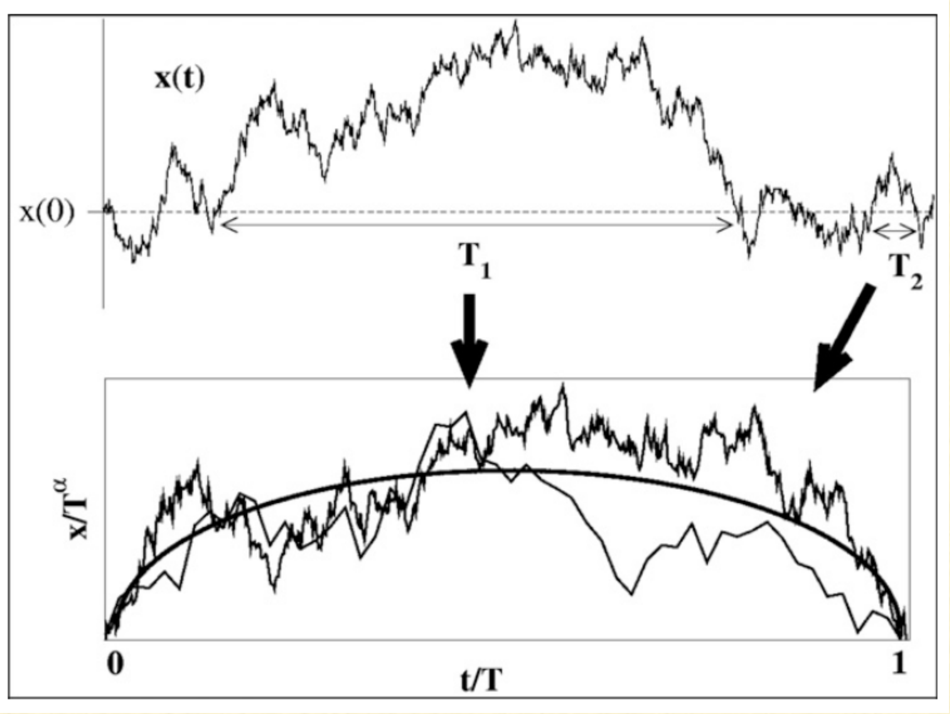
Average avalanche shape



Universality beyond exponents?

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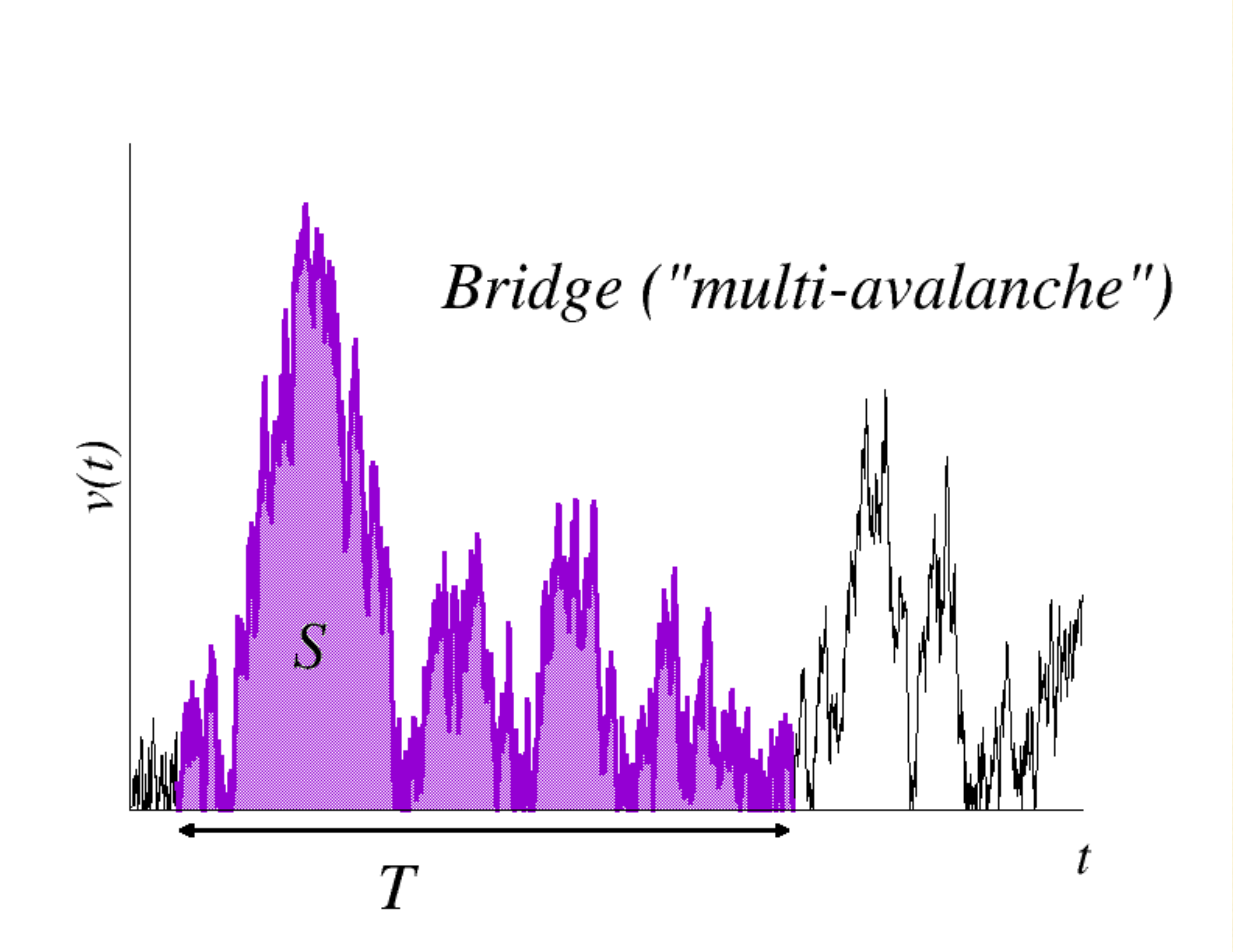
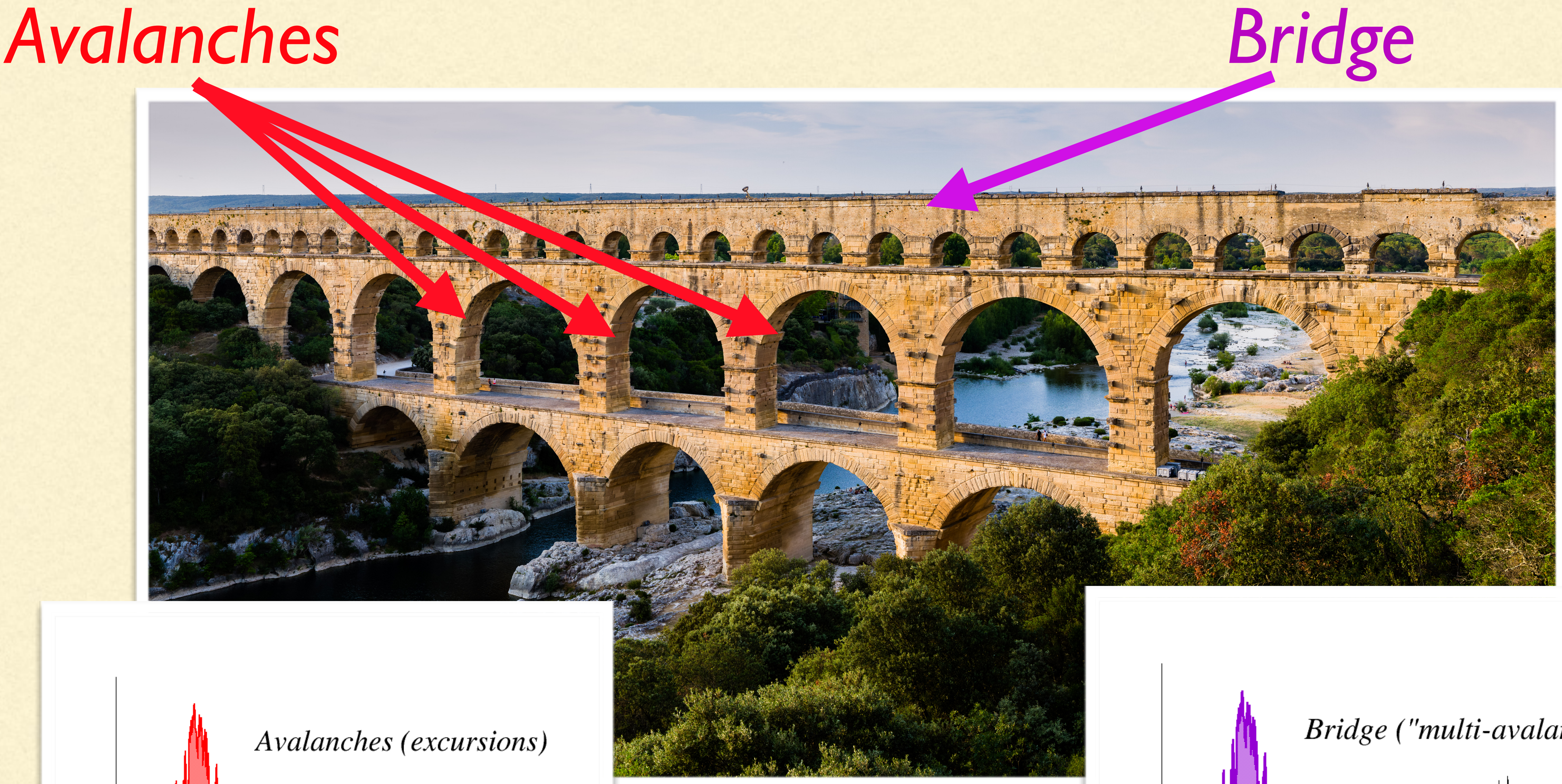
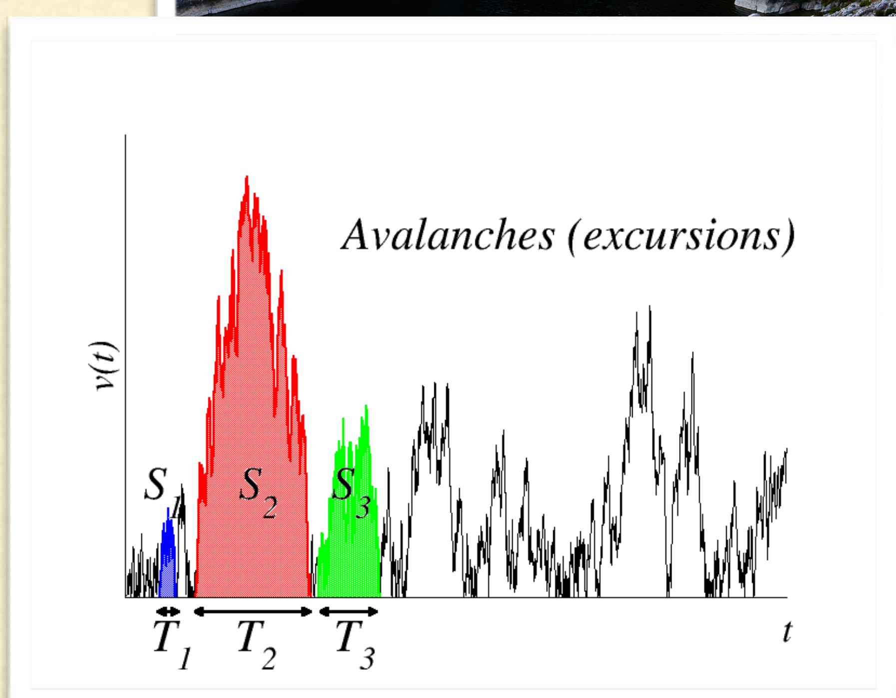
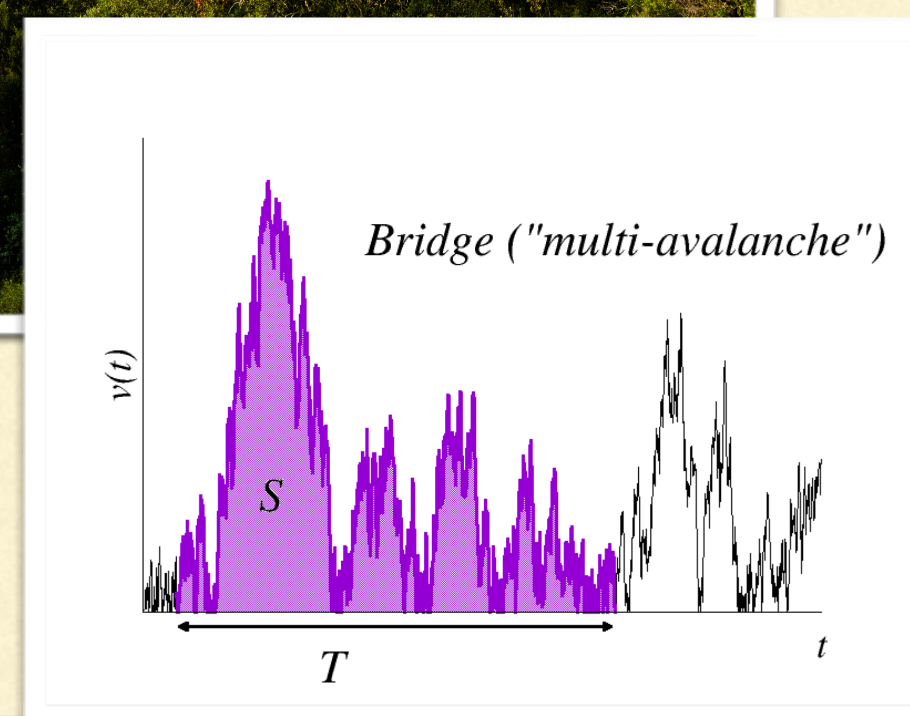
AVALANCHES AND BRIDGES



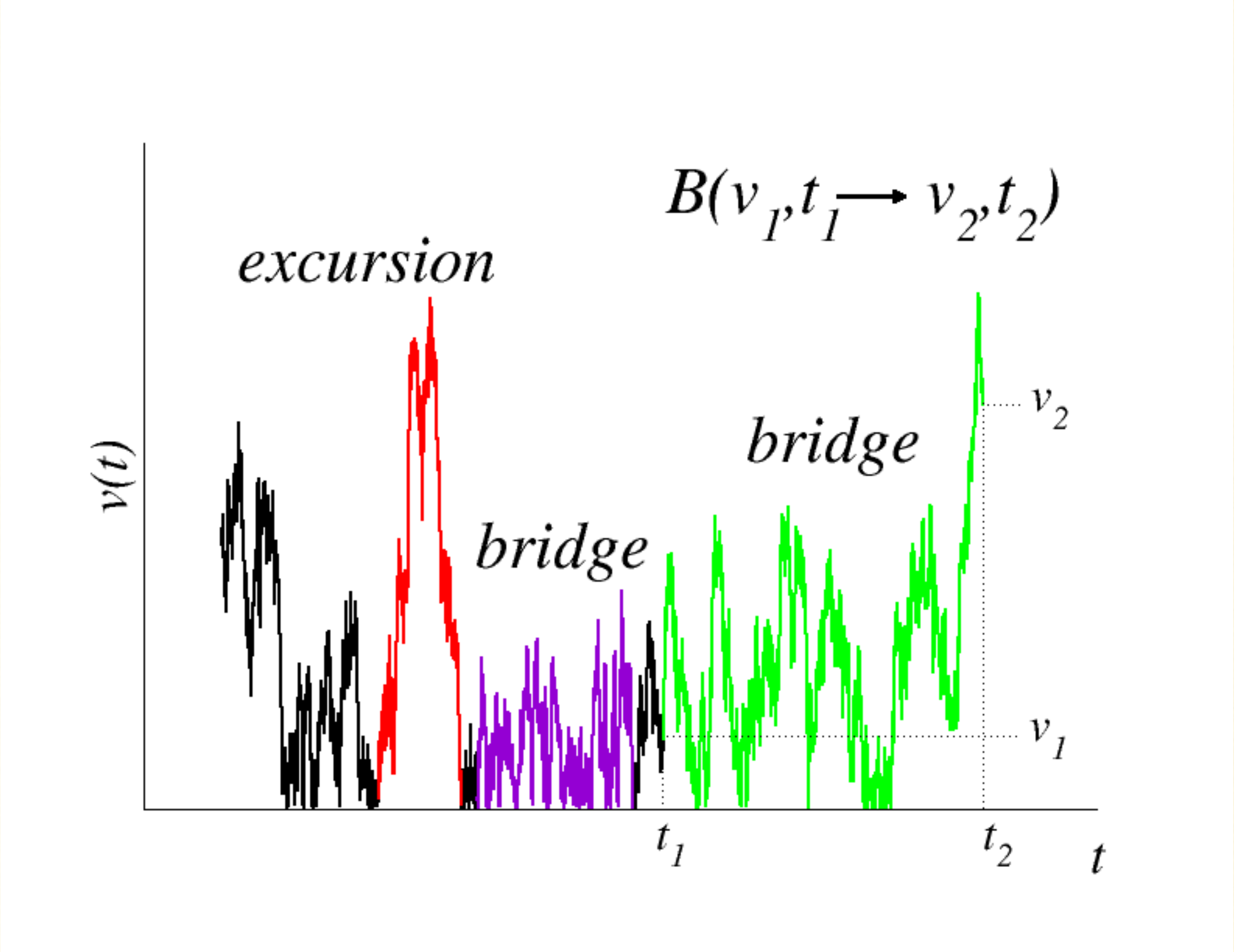
AVALANCHES AND BRIDGES







BRIDGE GENERAL DEFINITION



AVALANCHE VS. BRIDGE

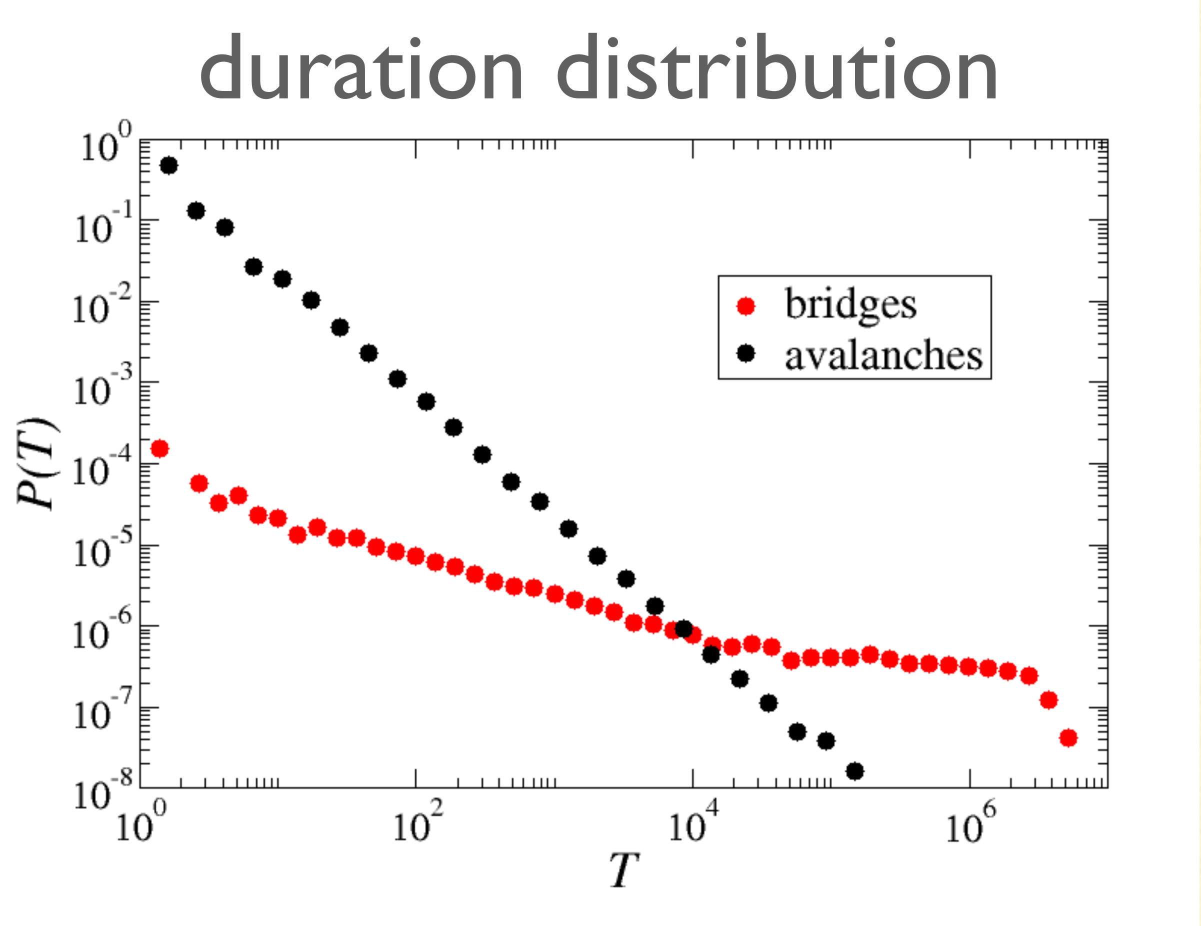
avalanche:

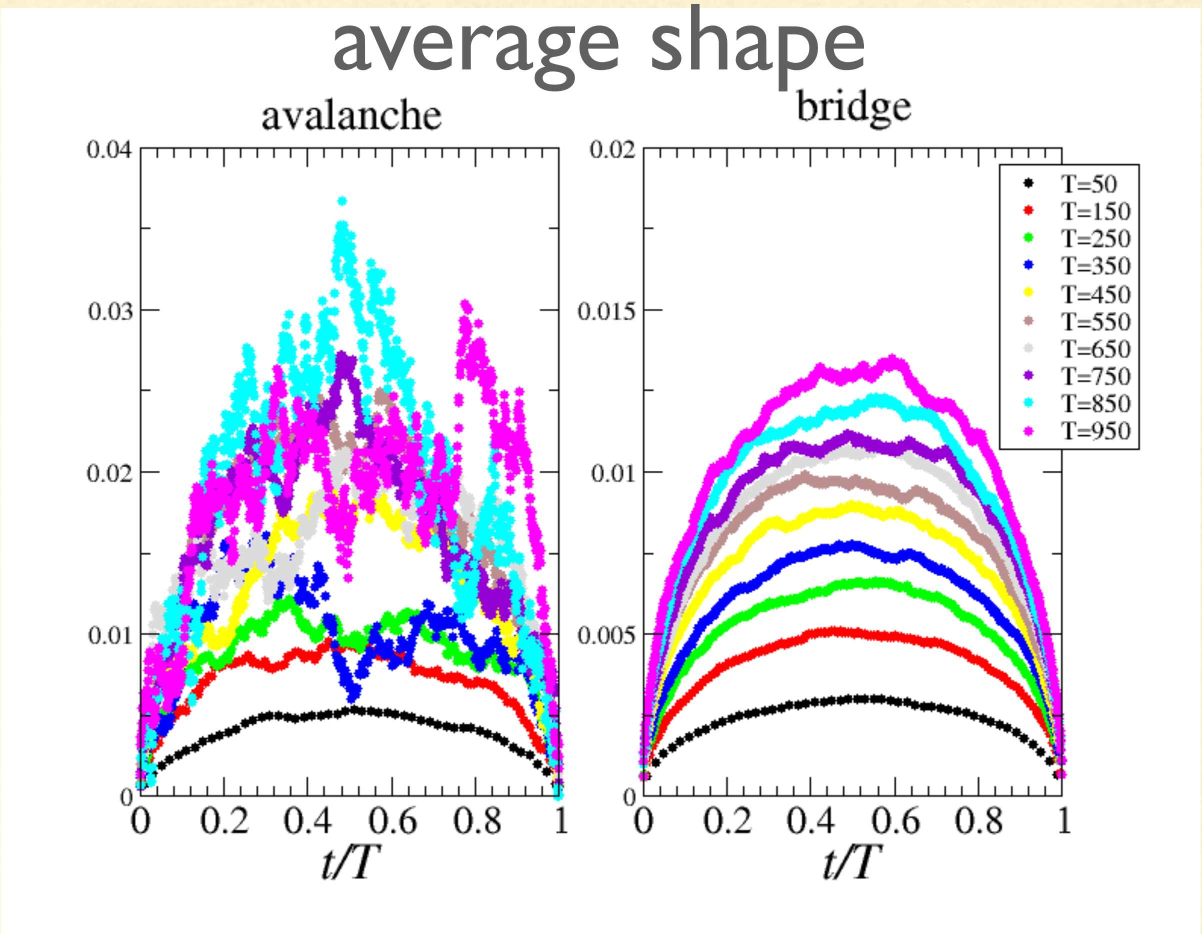
- constraint on all duration time
- poor statistics for large duration

bridge:

- constraint on initial and final time only
- large statistics for large duration

Example: numerical analysis stochastic signal (OU)

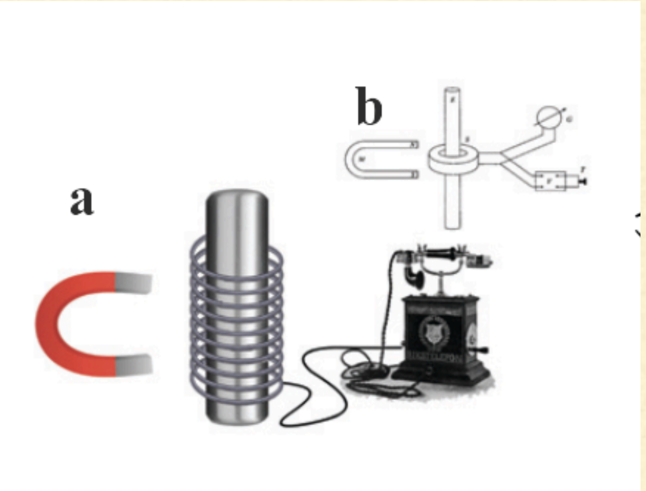


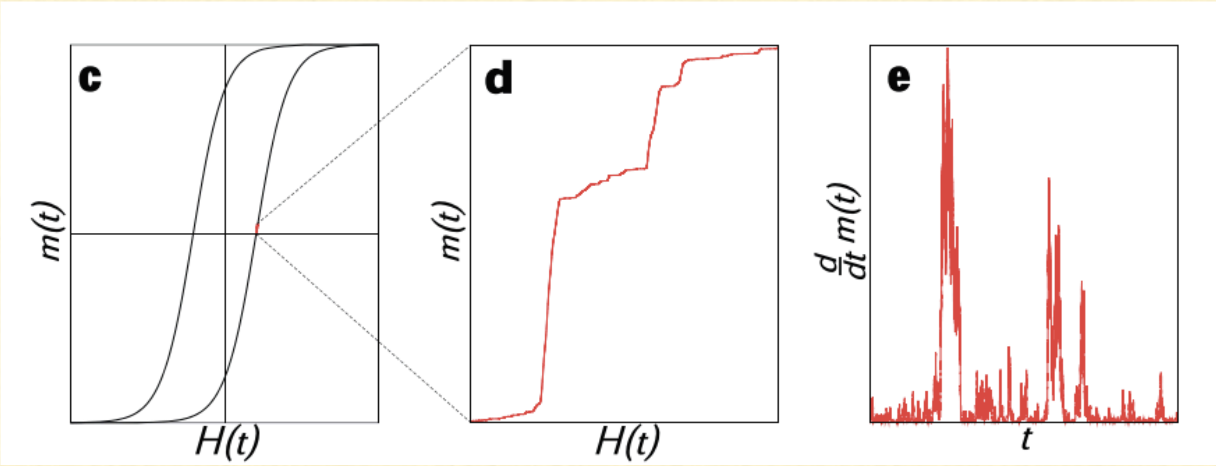


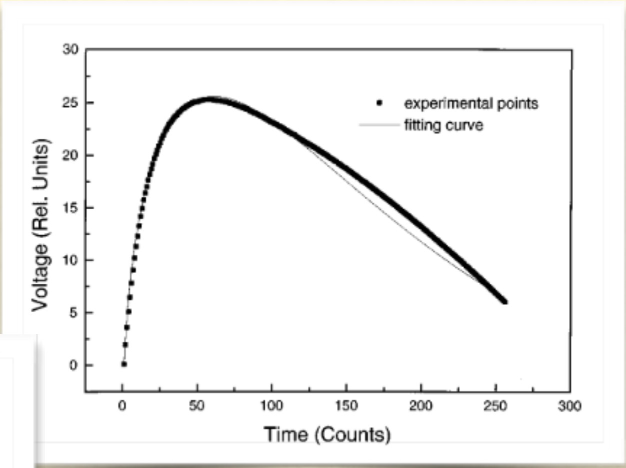
AVERAGE AVALANCHE SHAPE: EXPERIMENTAL OBSERVATIONS

A PORTFOLIO…

BARKHAUSEN NOISE







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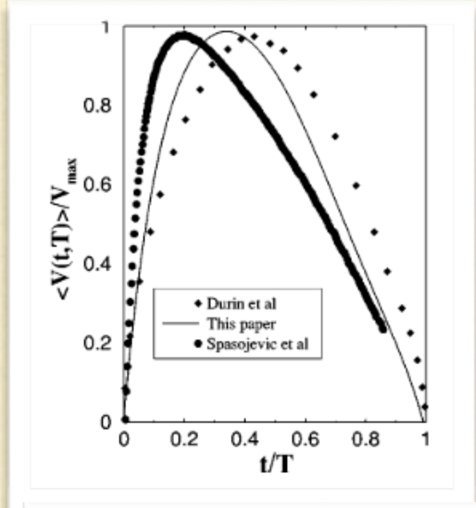

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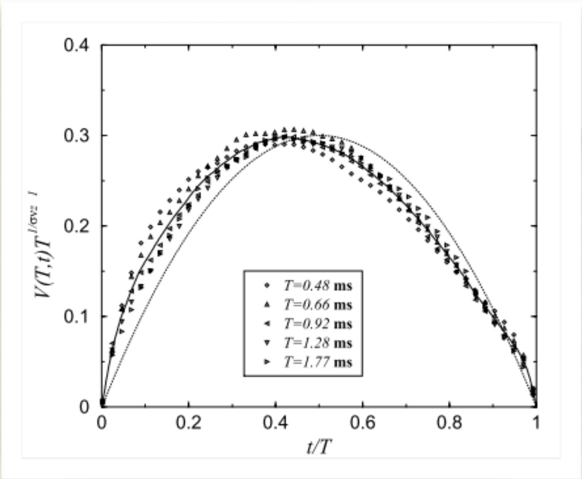

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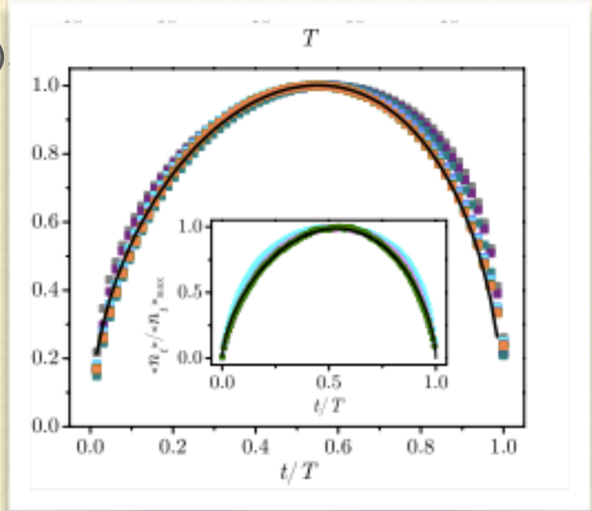

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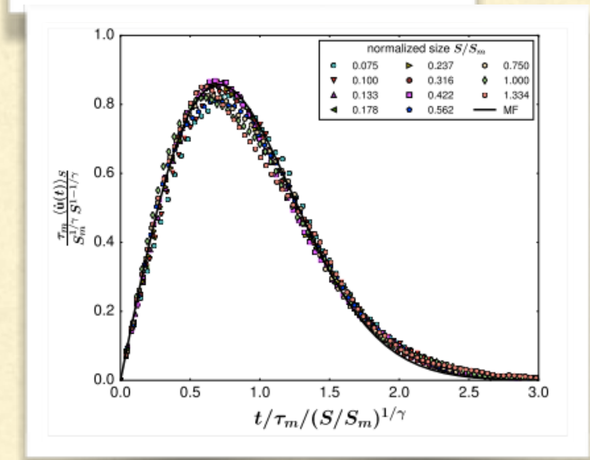

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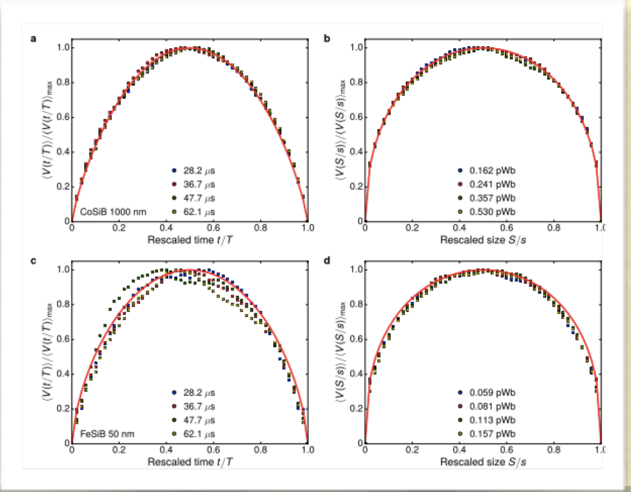

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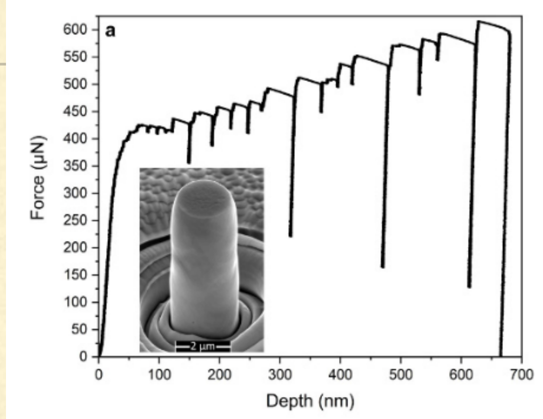

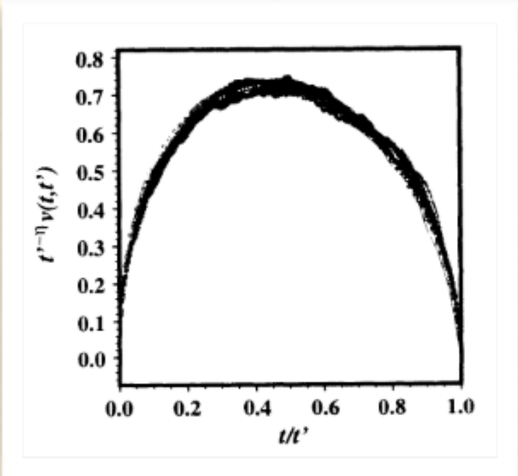

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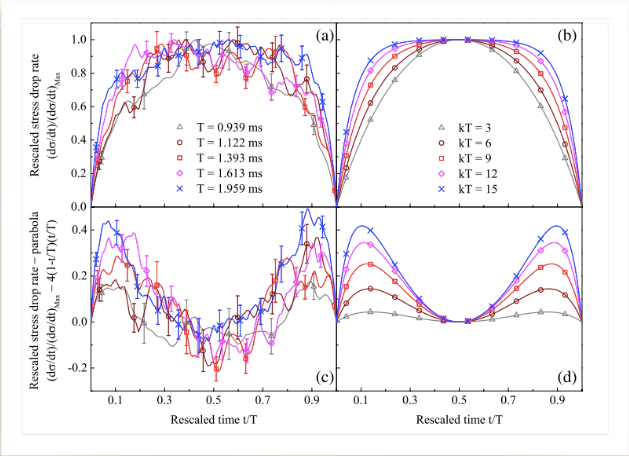

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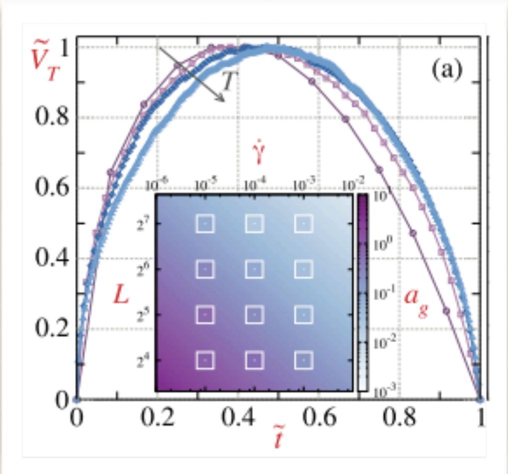

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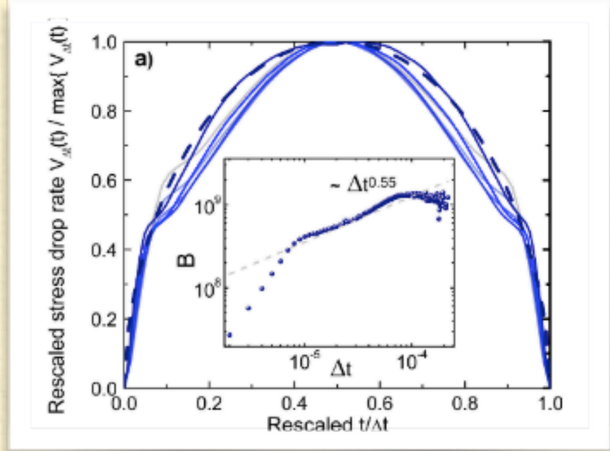

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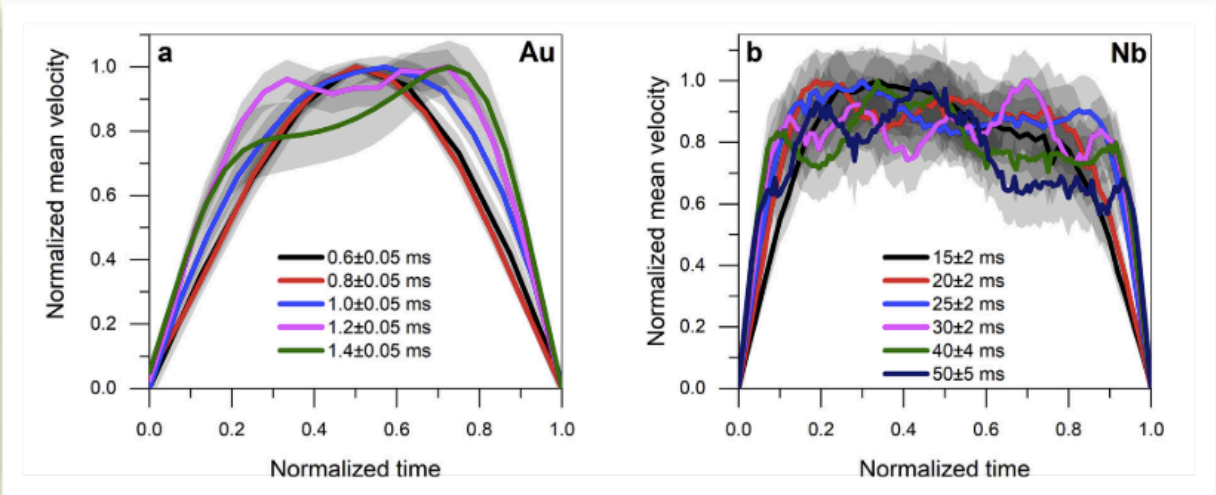

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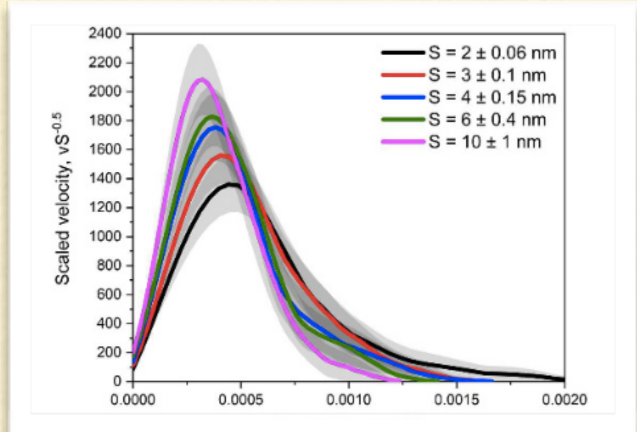

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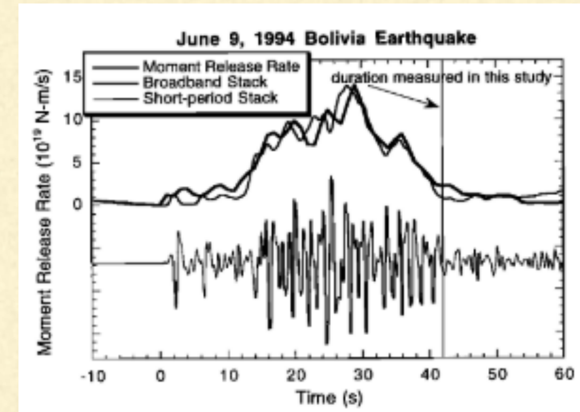

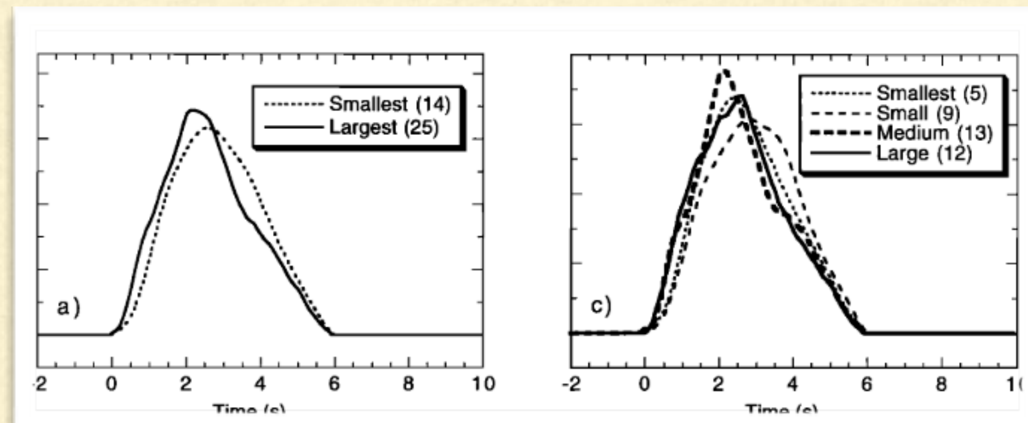

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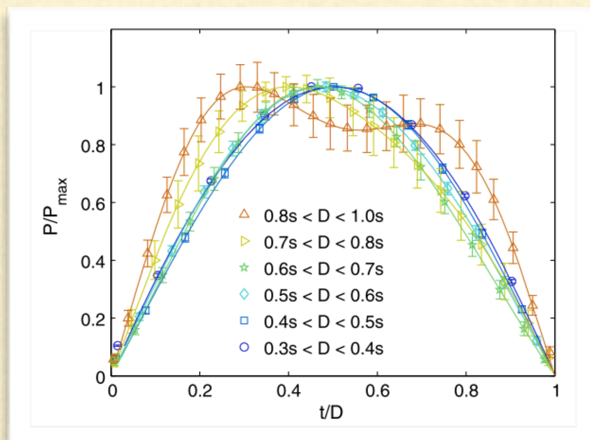

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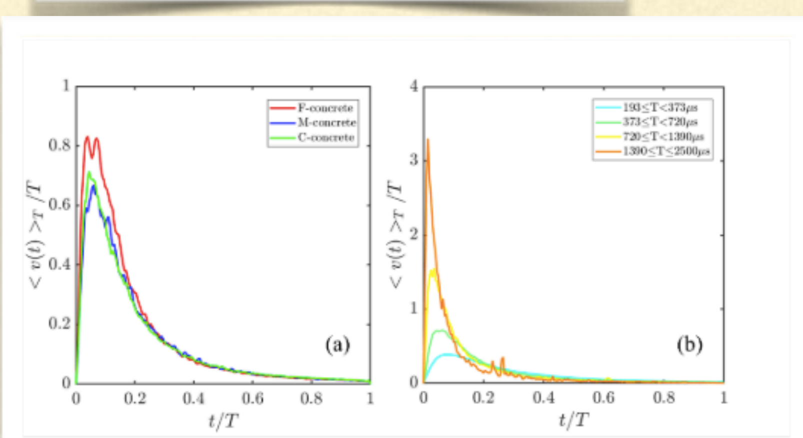

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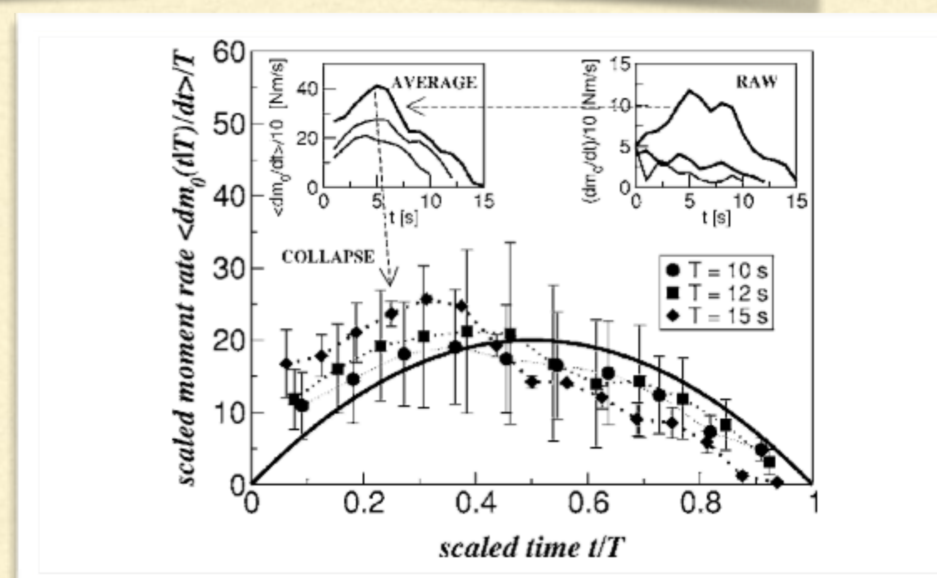

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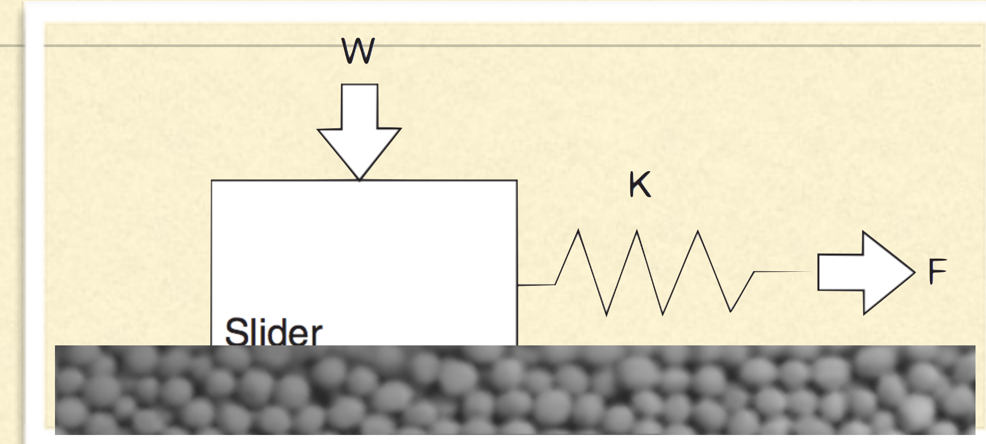

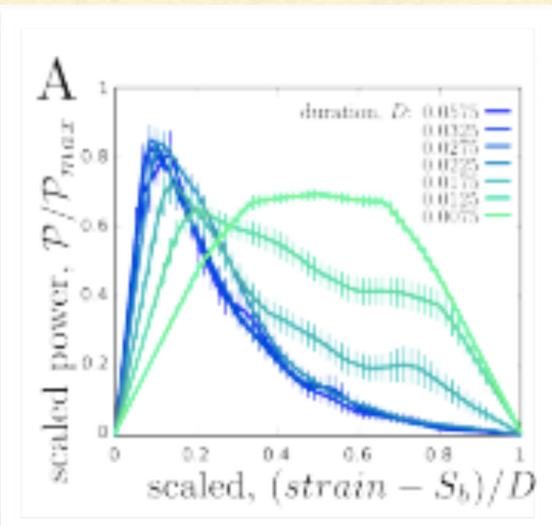

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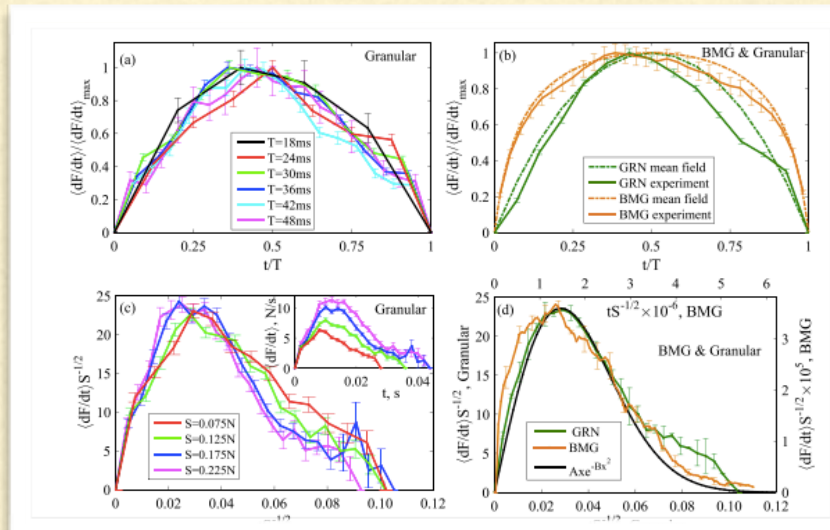

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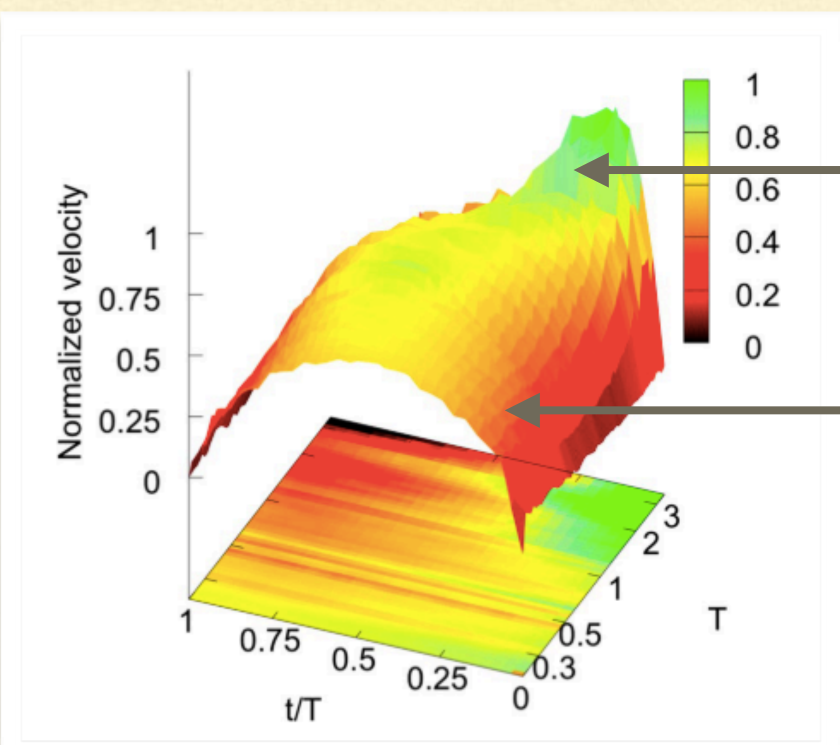

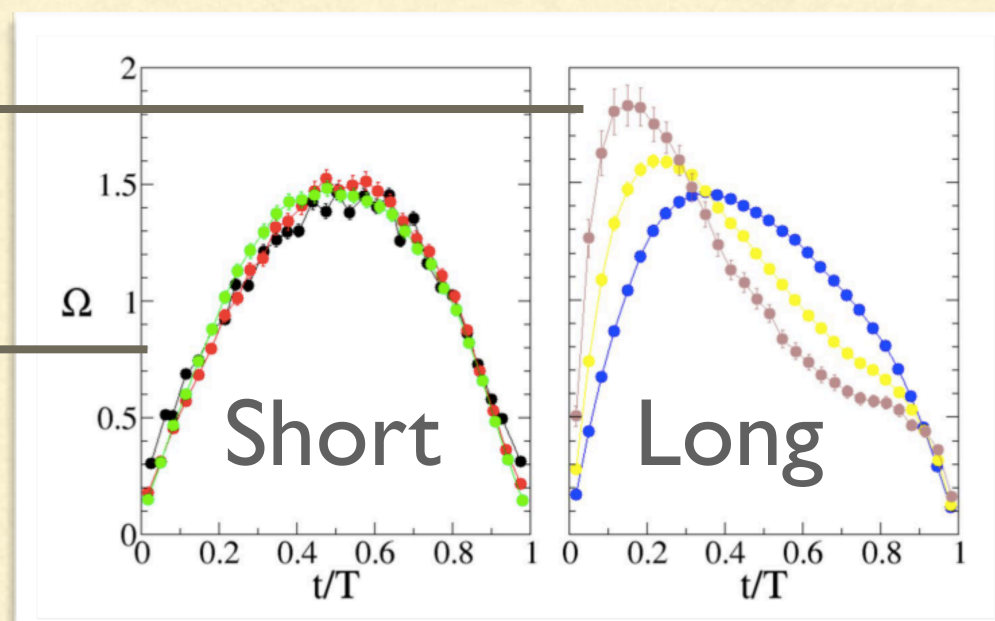

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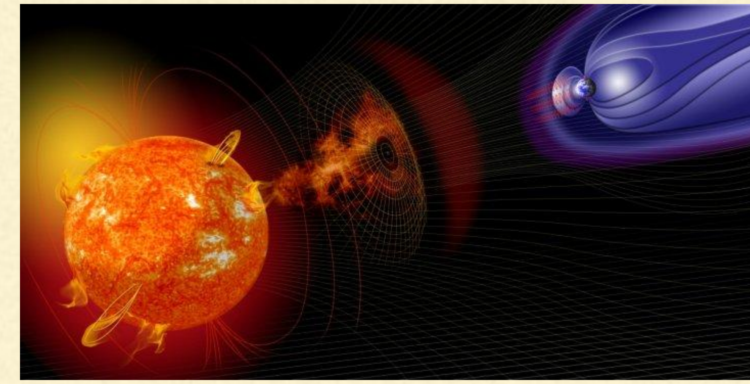
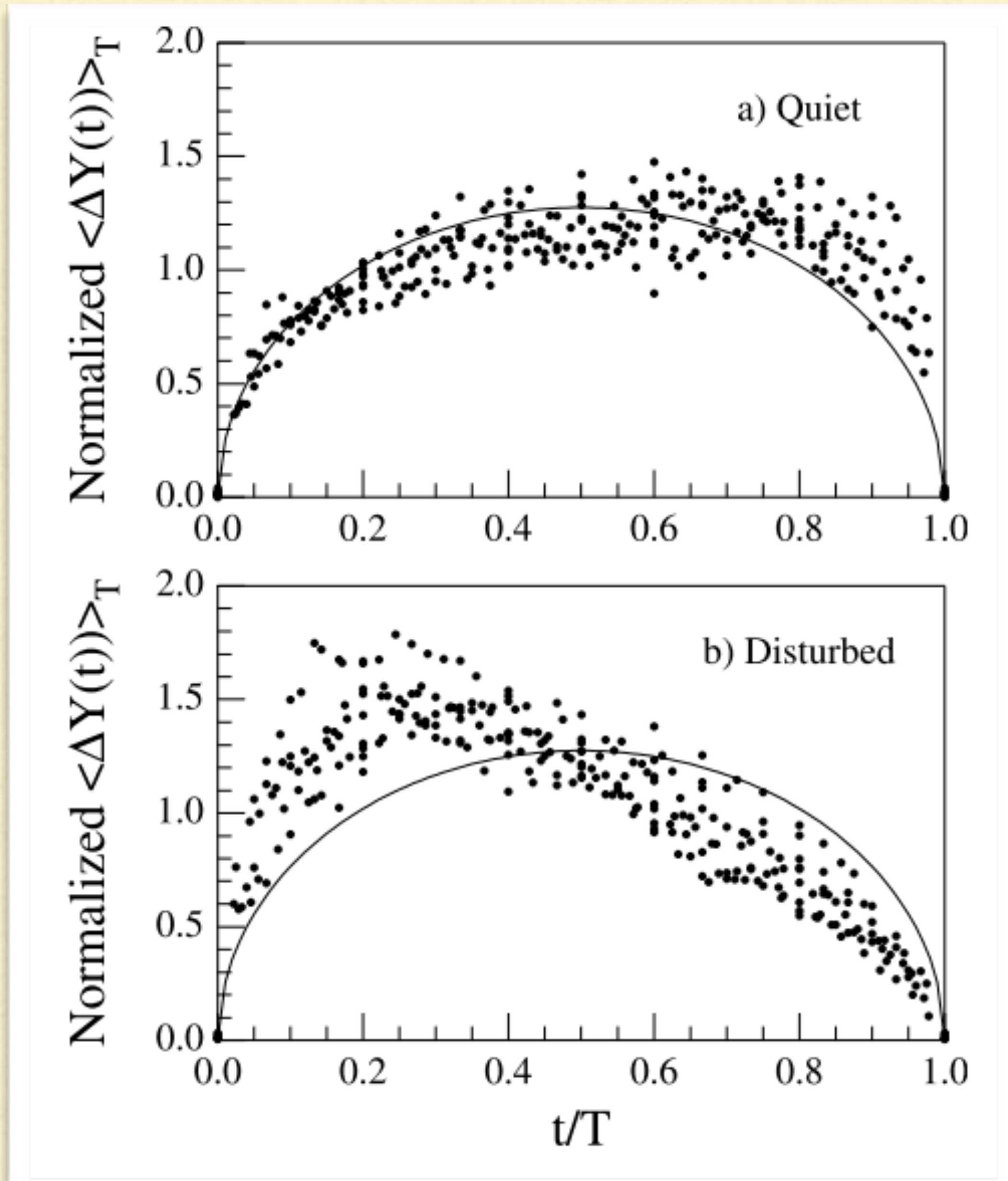
STARS AND PLANETS





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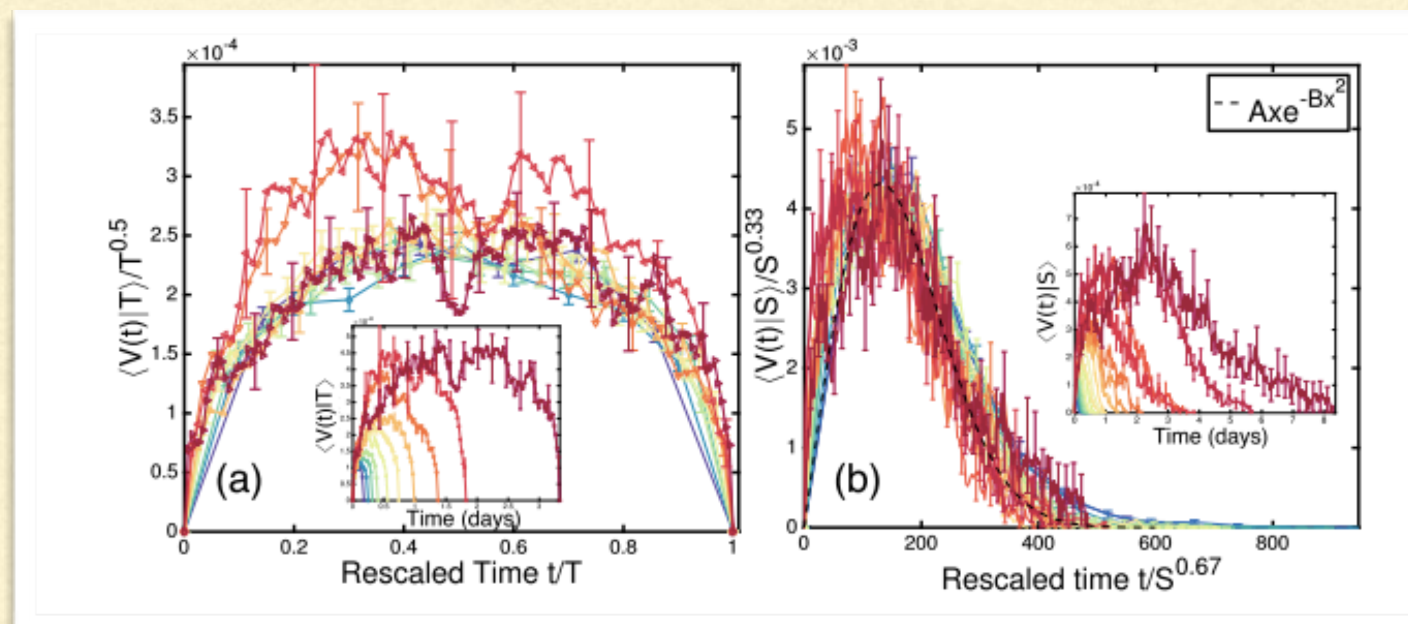

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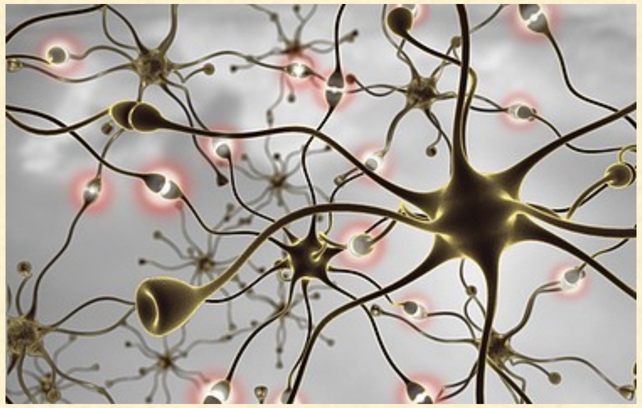

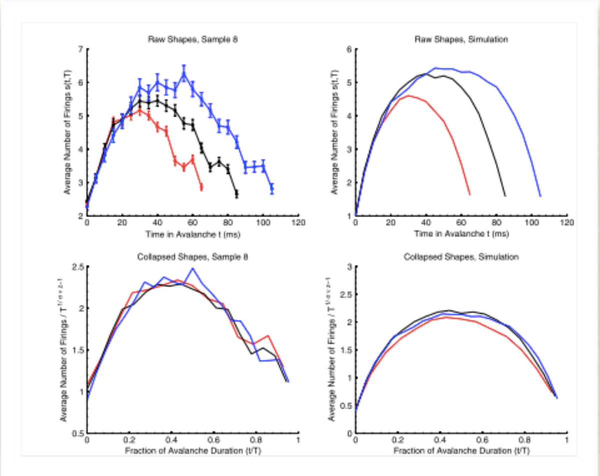

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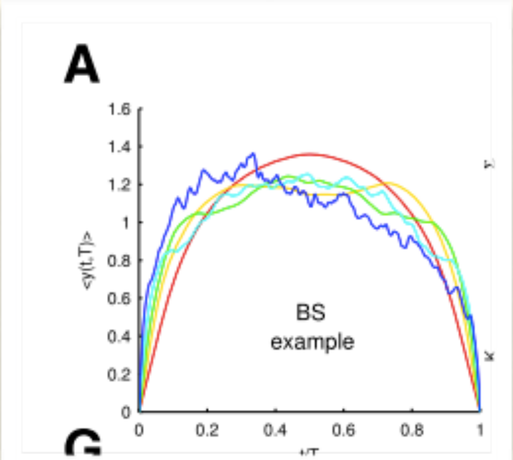

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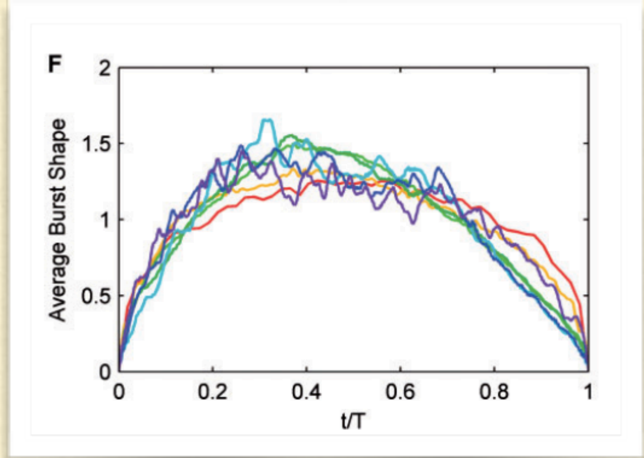

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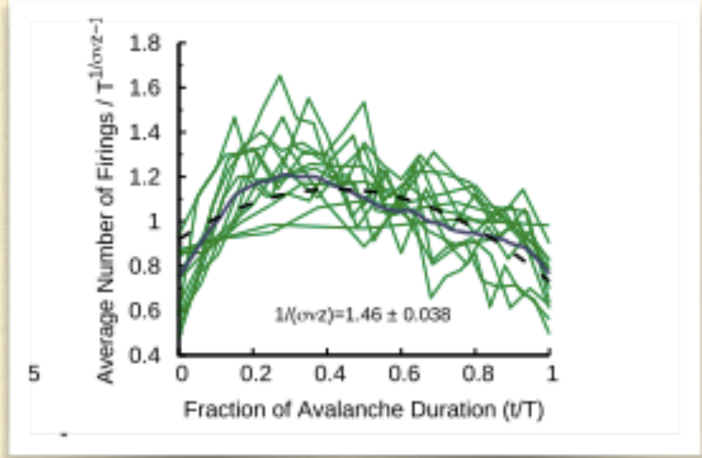

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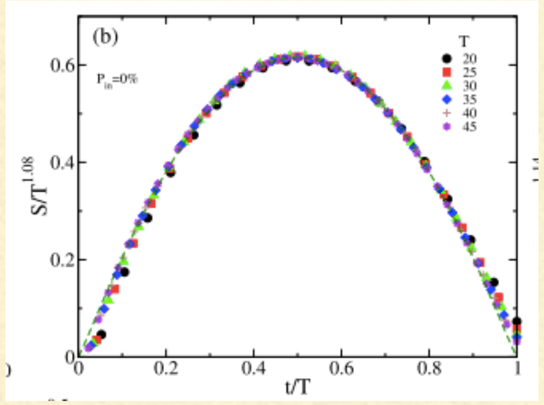

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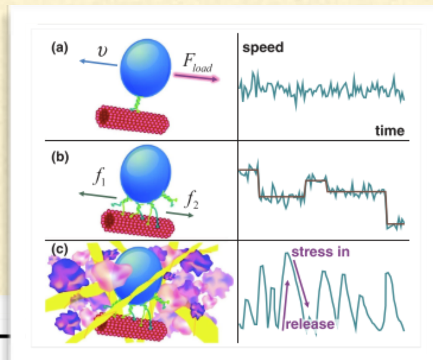
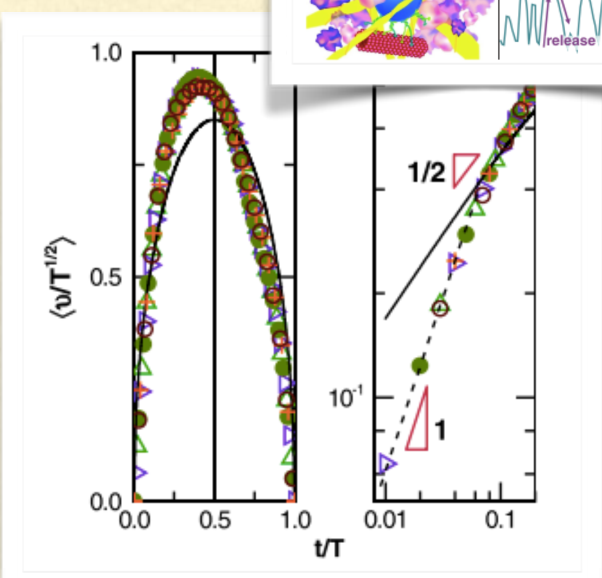
"ANIMATED" SYSTEMS





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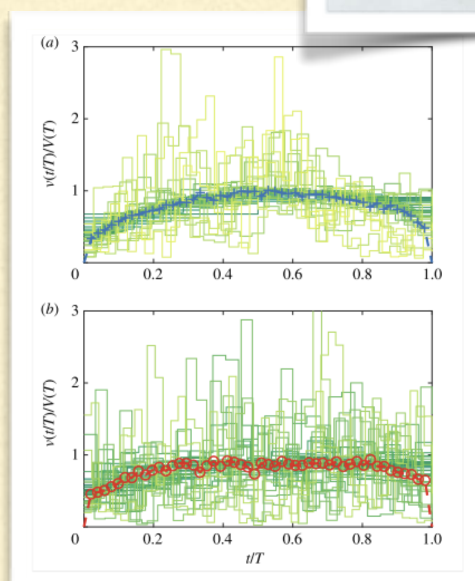

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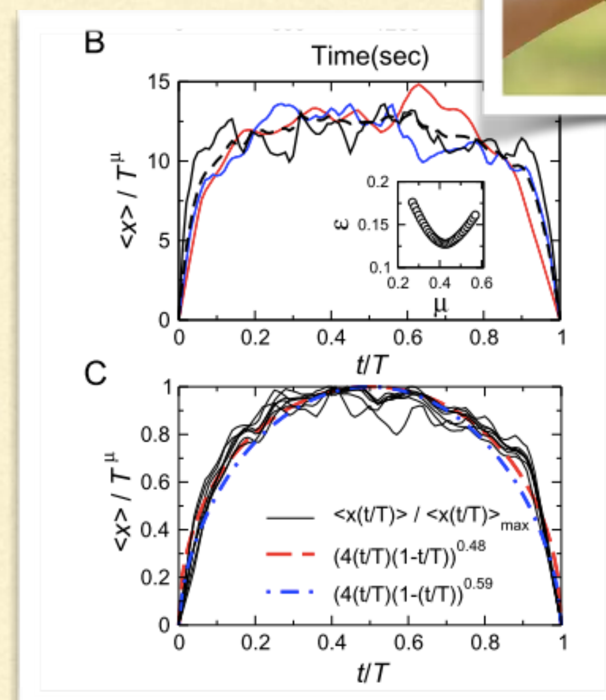

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ABBM MODEL EXACT SOLUTION

ABBM MODEL

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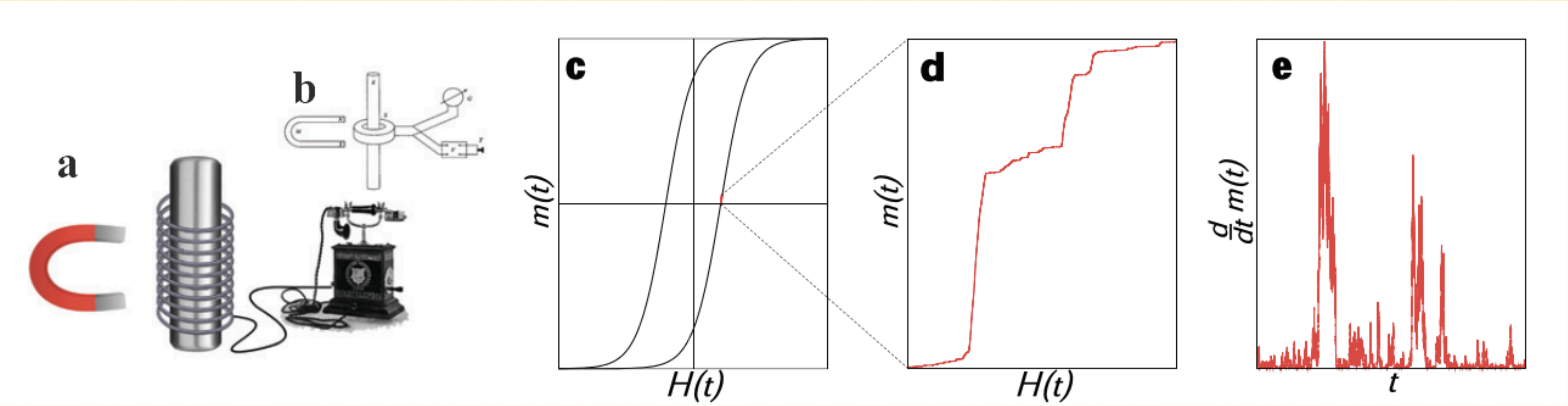
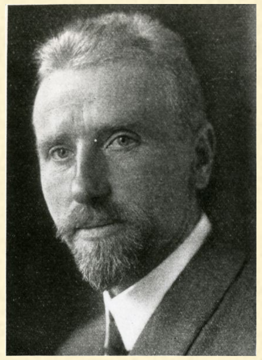
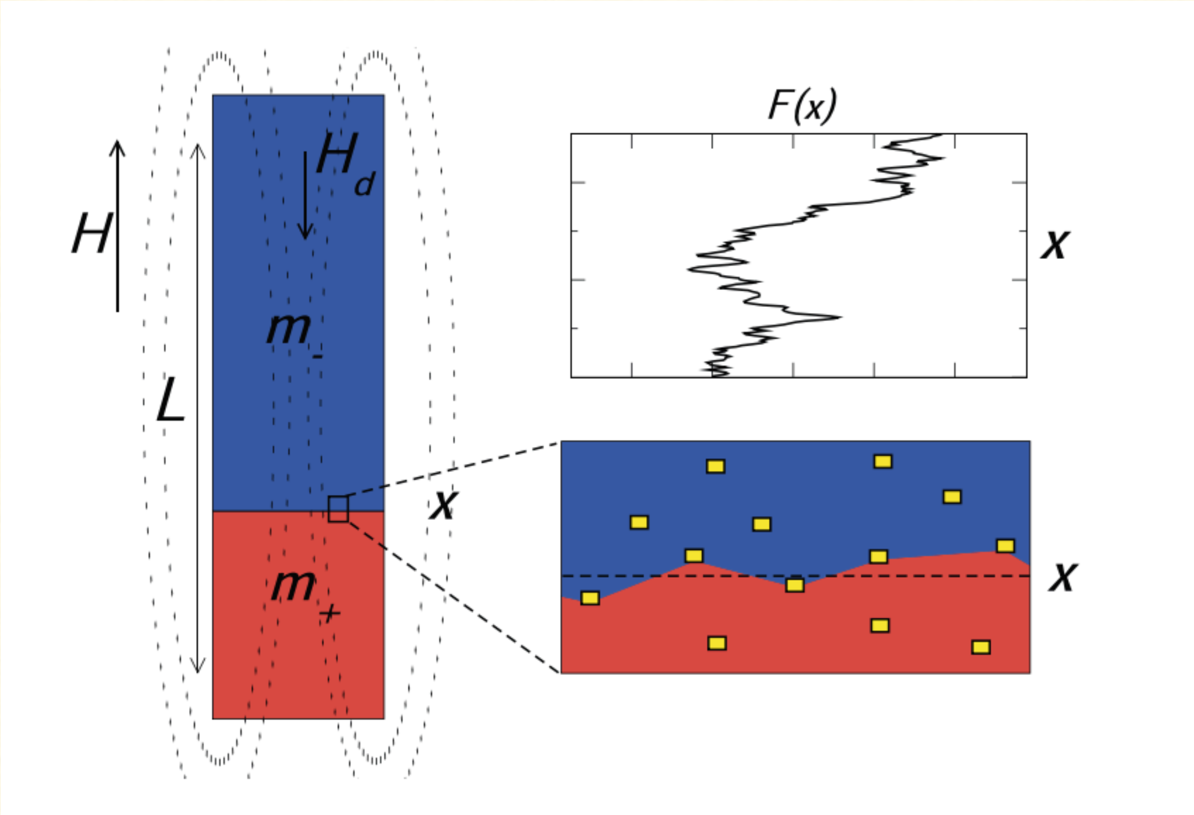




Barkhausen noise



over damped motion of domain wall

$$\frac{dx}{dt} = k\,(c\,t - x) + F(x)$$

drive *disorder*

"Brownian force"

$$F(x+dx) \approx F(x) \pm \sigma\sqrt{dx}$$

$$\frac{dF(x)}{dx} = \sigma\,\xi(x)$$

stochastic process in space

$$\overline{\xi(x)} = 0 \quad \overline{\xi(x)\xi(y)} = \delta(x-y)$$

How can we recast ABBM in a usual sde (with white noise "in time")?

TIME CHANGE

$$t \to \tau(t) \qquad \frac{d\tau}{dt} = G^2(X,t)$$

problem:

$$\frac{dX(t)}{dt} = \mu(X,t) + \sigma(X,t)\xi(t) \xrightarrow{?} \frac{dX(\tau)}{d\tau} = \hat{\mu}(X,\tau) + \hat{\sigma}(X,\tau)\xi(\tau)$$

solution:

$$\hat{\mu}(X,\tau) = \frac{\mu(X,t(\tau))}{G^2(X,t(\tau))} \qquad \hat{\sigma}(X,\tau) = \frac{\sigma(X,t(\tau))}{G(X,t(\tau))}$$

example of (deterministic) time change:

Brownian process

$$\frac{dx(t)}{dt} = \xi(t)$$

time change

$$t(\tau) = \frac{\sigma^2}{2k}\left(e^{2k\tau} - 1\right)$$

new (rescaled) process

$$y(\tau) = e^{-k\tau} x\left(t(\tau)\right)$$

Ornstein-Uhlenbeck process

$$\frac{y(\tau)}{d\tau} = -k\, y(\tau) + \sigma\xi(\tau)$$

TIME CHANGE: FROM ABBM TO CIR

ABBM $v(t) = k\,(c\,t - x) + F(x)$ $\quad \boxed{\dfrac{dF(x)}{dx} = \sigma\,\xi(x)}$

$$\overline{\xi(x)} = 0$$
$$\overline{\xi(x)\xi(y)} = \delta(x-y)$$

ABBM in space coordinates

white noise in space

$$\frac{dv(t)}{dt} = k\,(c-v) + \frac{dF(x)}{dt} \qquad \frac{dF(x)}{dt} = \frac{dF(x)}{dx}\frac{dx}{dt} = v\sigma\xi(x)$$

$$\frac{dv}{dx} = k\,\left(\frac{c}{v(x)} - 1\right) + \sigma\xi(x)$$ *Rayleigh process (in space)*

time change $x \to t$ $\qquad \dfrac{dx}{dt} = v(t) \equiv G^2(t) \geq 0$

Cox-Ingersoll-Ross process (**CIR**)

$$\frac{dv(t)}{dt} = k\,\left(c - v\right) + \sigma\sqrt{v}\,\xi(t)$$

$$\overline{\xi(t)} = 0$$
$$\overline{\xi(t)\xi(s)} = \delta(t-s)$$

white noise in time

J. Cox, J. E. Ingersoll, and S. A. Ross, Econometrica (1985).

CONTINOUS MARKOV PROCESSES

$$P(x_0, t_0; \ldots; x_n, t_n) = P(x_0, t_0) P(x_0 \xrightarrow{t_1 - t_0} x_1) \ldots P(x_{n-1} \xrightarrow{t_n - t_{n-1}} x_n)$$

$P(\cdot \to \cdot)$ propagator

FP: $\partial_t P = \partial_x \left[-\mu(x,t) P + \frac{1}{2} \partial_x \left(\sigma^2(x,t) P \right) \right]$

sde: $\frac{dx}{dt} = \mu(x,t) + \sigma(x,t)\, \xi(t)$

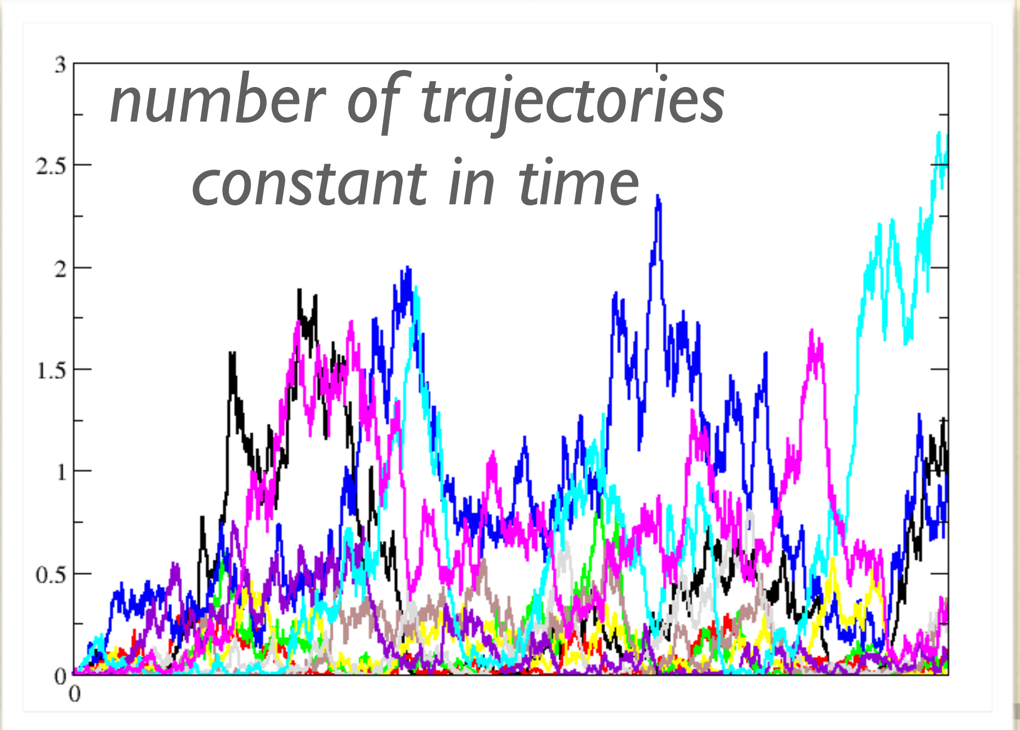
solution depends on b.c.

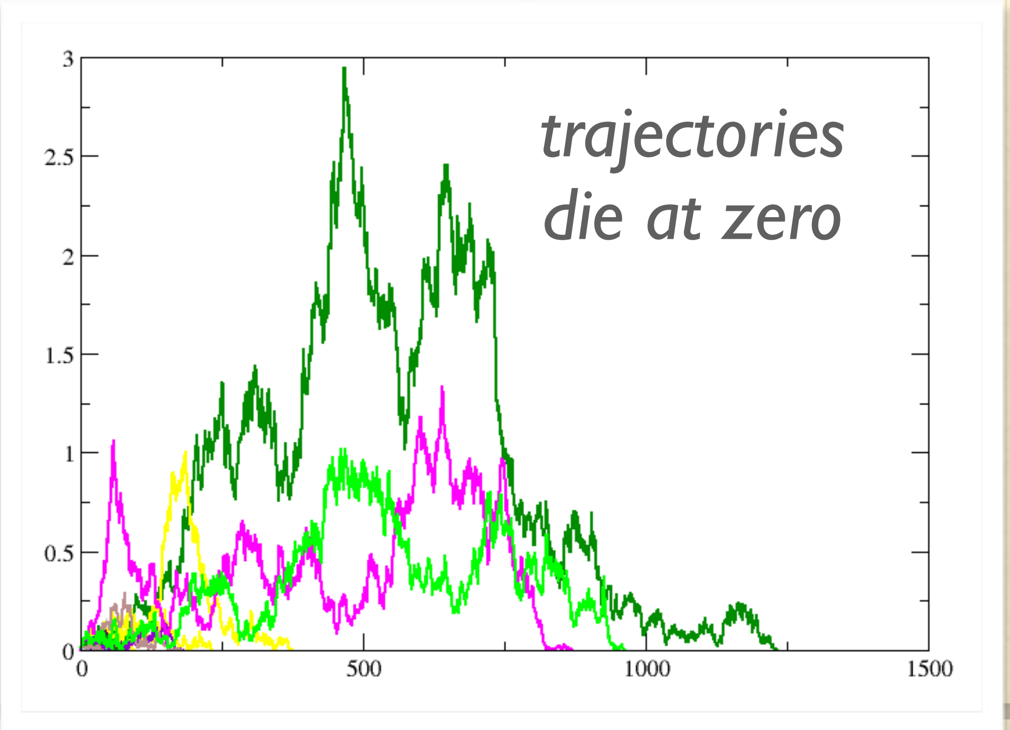
Notable cases:

"free" $\int P(x_0 \xrightarrow{t} x)\, dx = 1$

"absorbing" $\frac{d}{dt} \int P(x_0 \xrightarrow{t} x) dx < 0$

$P(0 \xrightarrow{t} v) = 0$





FELLER'S SOLUTION OF ABBM/CIR FP (1951)

$k = 1$
$\sigma = \sqrt{2}$

$$\frac{dv}{dt} = (c - v) + \sqrt{2v}\,\xi(t) \qquad \partial_t P(v,t) = \partial_v^2\left[vP(v,t)\right] - \partial_v\left[(c-v)P(v,t)\right]$$

Possible solutions depend on the value of c (drive rate)

$c > 1$

fast drive: "steady sliding"

only "free" solution

zero *unattainable*

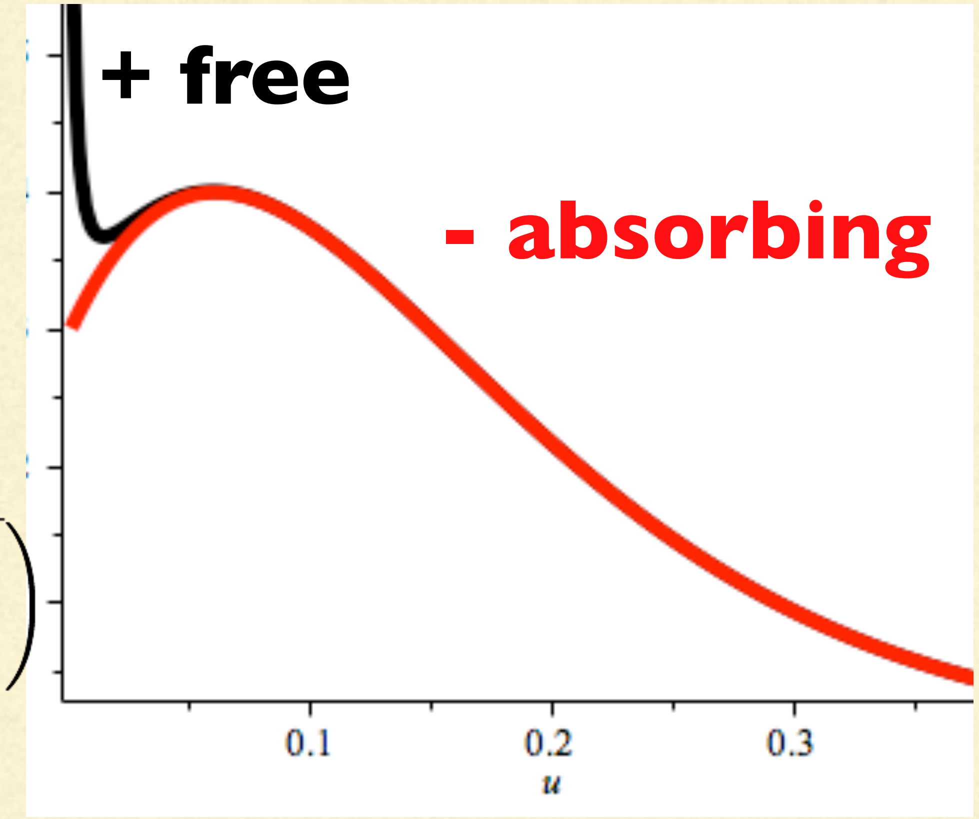
$c < 1$

slow drive: "stick-slip"

1. "free" solution
2. "absorbing" solution

zero *attainable*

$P(v_0 \xrightarrow{t} v)$



$$P_\pm(v_0 \xrightarrow{t} v) = \frac{\exp\left(-\frac{v+v_0e^{-t}}{1-e^{-t}}\right)}{1-e^{-t}} \left(\frac{v}{v_0e^{-t}}\right)^{c-1} I_{\pm(c-1)}\left(\sqrt{\frac{v\,v_0e^{-t}}{(1-e^{-t})^2}}\right)$$

$I_q(z)$ mod. Bessel 1st kind

+ free

$$P(v_0 \xrightarrow{t} v) \propto v^{c-1} \text{ for } v \approx 0$$

- absorbing

$$P(v_0 \xrightarrow{t} v) \approx \text{ const. for } v \approx 0$$

ABBM AVERAGE AVALANCHE & BRIDGE SHAPE

ABBM AVERAGE SHAPE: PREVIOUS WORKS

Approximation: $\langle v(x)\rangle_S \xrightarrow{x\approx\langle x(t)\rangle, S\approx\langle S(T)\rangle} \langle v(t)\rangle_T$

F. Colaiori, Adv. Phys.(2008).

sinusoidal shape: $\langle v(t)\rangle_T = \frac{\pi T}{2}\sin\left(\pi\frac{t}{T}\right)$

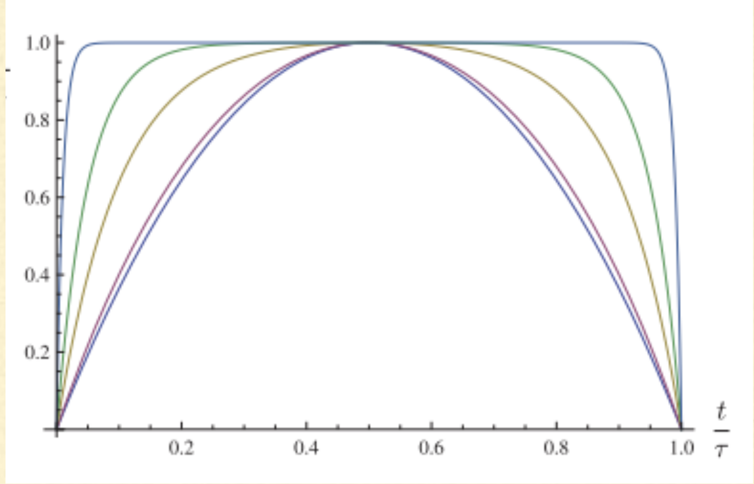
Exact ABBM for vanishing drive $c \to 0$

S. Papanikolaou, F. Bohn, R. L. Sommer, G. Durin, S. Zapperi, and J. P. Sethna, Nat. Phys. 7, 316 (2011).

$$\langle v(t)\rangle_T = \frac{1}{2k}\frac{\left(e^{2k(T-t)}-1\right)\left(e^{2kt}-1\right)}{e^{2kT}-1}$$

parabolic shape in the scaling regime $T \ll \frac{1}{k}$

$$\langle v(t)\rangle_T = \frac{1}{8}t\left(1-\frac{t}{T}\right)$$



Same result from theory of disordered elastic manifolds for $d \geq d_{uc}$

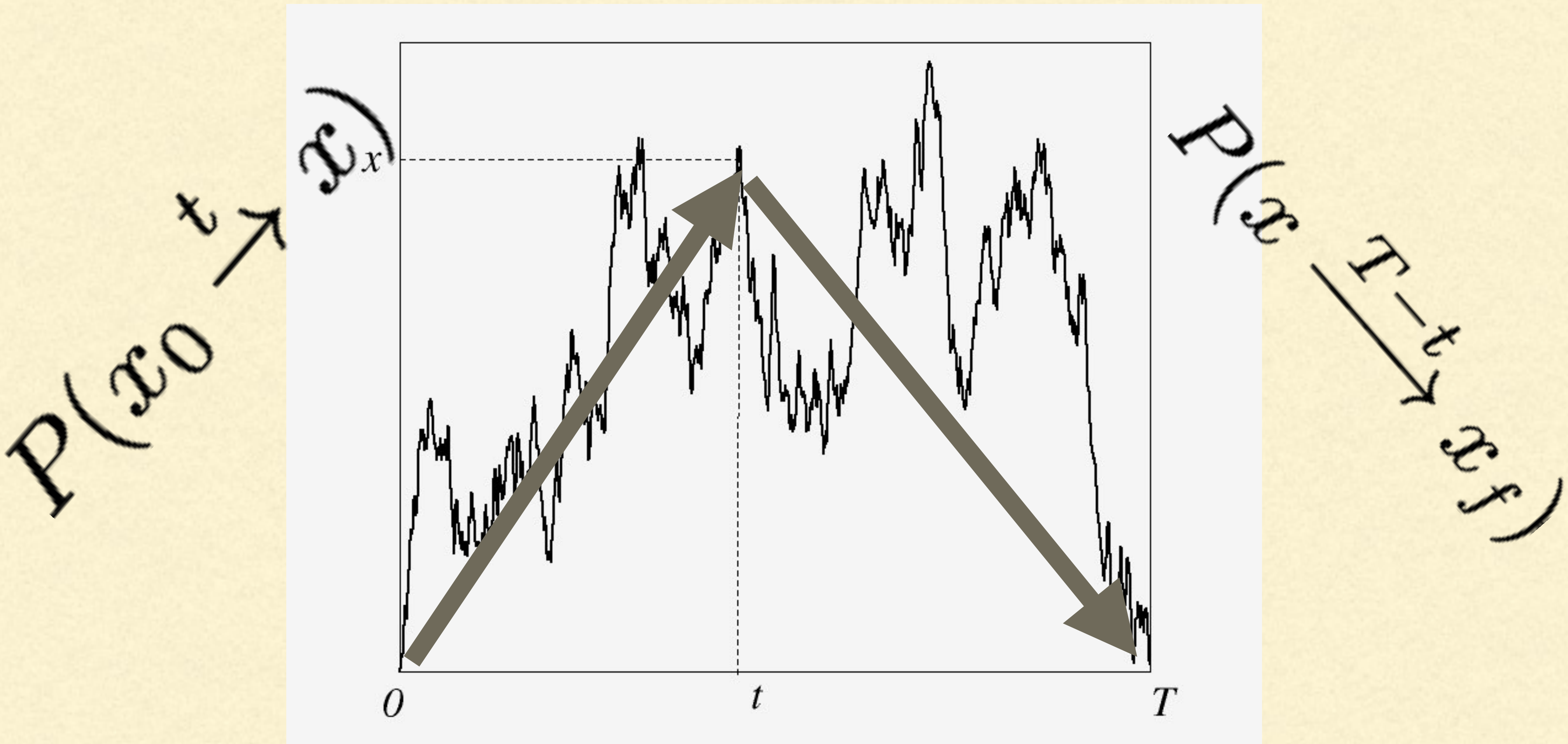
$$\left(\eta_0\partial_t - \nabla_x^2\right)\dot{u}(x,t) = \partial_t F\left(ct + u(x,t), x\right) - k\,\dot{u}(x,t)$$

$$v(t) = L^{-d}\int dx\,\dot{u}(x,t)$$

P. Le Doussal and K. J. Wiese, Eur. Lett. (2012)
K. J. Wiese, *arXiv :2102.01215, ROPP* (2022).

AVALANCHE/BRIDGE DISTRIBUTION FOR MARKOV PROCESSES

$$P(x_0,t_0;\,...;x_n,t_n) = P(x_0,t_0)P(x_0 \xrightarrow{t_1-t_0} x_1)\,...\,P(x_{n-1} \xrightarrow{t_n-t_{n-1}} x_n)$$



Bridge:

$$B(x,t|x_0 \xrightarrow{T} x_f) = \frac{P(x_0 \xrightarrow{t} x)\,P(x \xrightarrow{T-t} x_f)}{P(x_0 \xrightarrow{T} x_f)}$$

"free" propagator

normalization

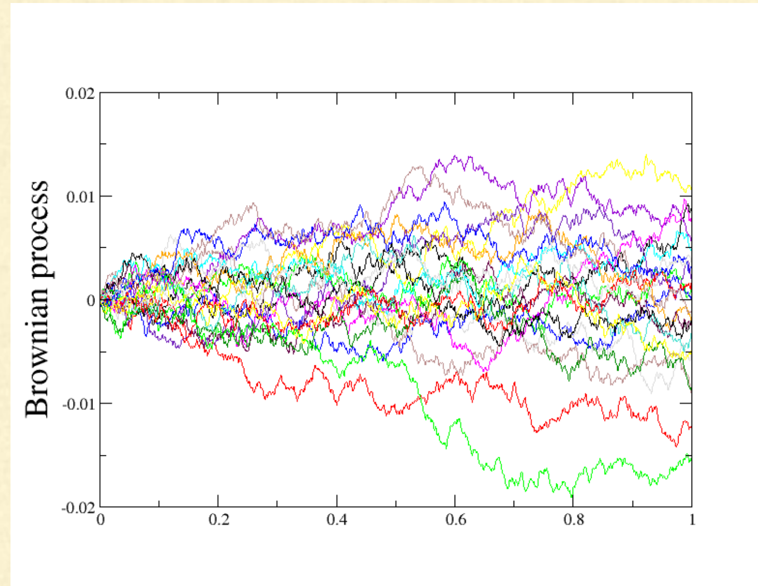
Avalanche:

same expression with "absorbing" propagator

SDE FOR CONSTRAINED BROWNIAN PROCESS

Brownian process

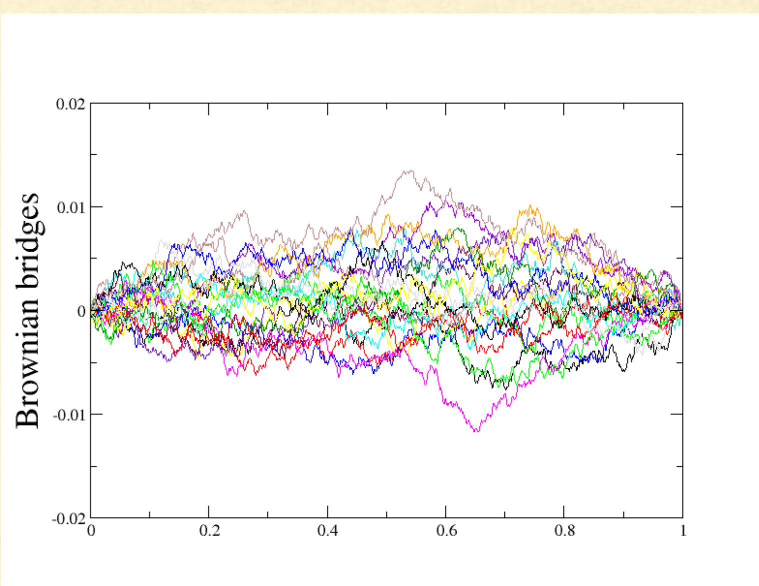
$$\frac{dv}{dt} = \xi(t)$$



S. N. Majumdar and H. Orland, J. Stat. Mech. (2015).
J. Pitman and M. Yor, A Guide to Brownian Motion and Related Stochastic Processes, (2018).
A. Mazzolo, J. Stat. Mech.(2017).

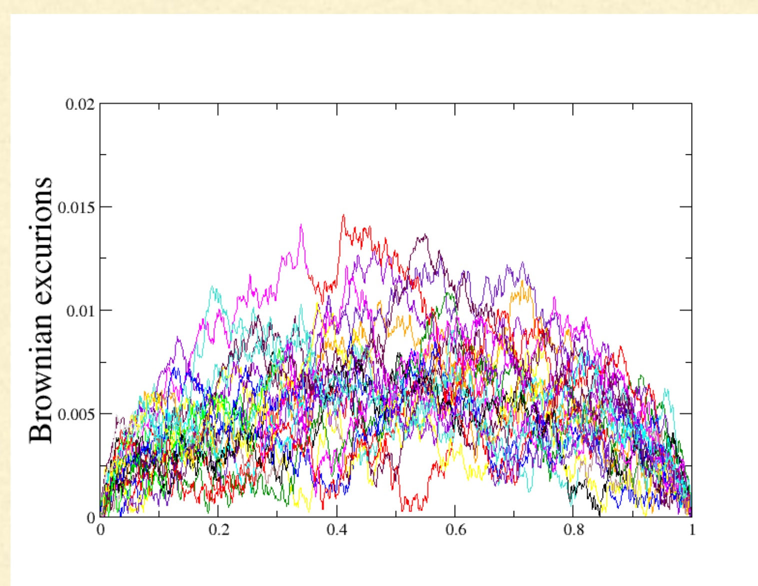
Brownian Bridge

$$\frac{dv}{dt} = -\frac{v}{T-t} + \xi(t)$$



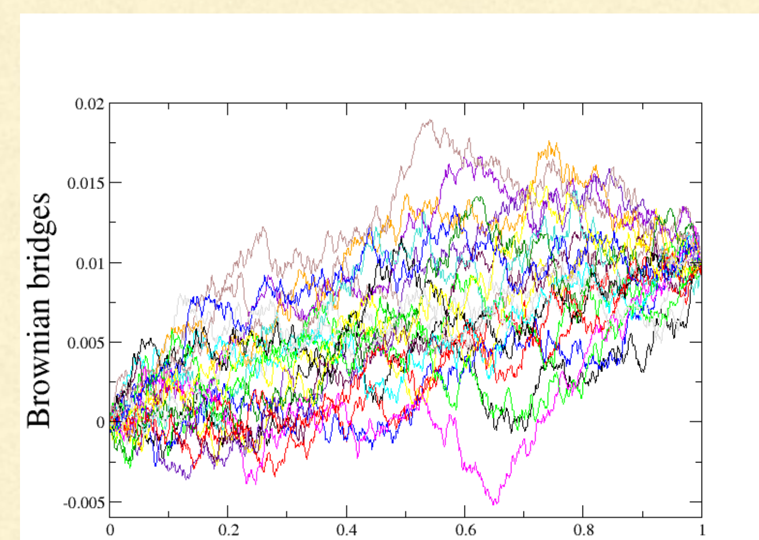
Brownian Excursion

$$\frac{dv}{dt} = \frac{1}{v} - \frac{v}{T-t} + \xi(t)$$



Brown. Bridge (general)

$$\frac{dv}{dt} = \frac{v_f - v}{T-t} + \xi(t)$$



$$B(0, 0 \to v_f, T)$$

DOOB'S TRANSFORM

$$\frac{dv}{dt} = \mu(v) + \sigma(v)\xi(t)$$

sde unconstrained process

Process constrained to $v(T) = v_f$

$$\frac{dv}{dt} = \mu(v) + \sigma^2(v)\frac{d}{dv}P(v, t \to v_f, T) + \sigma(v)\xi(t)$$

if P is:

- the "free" propagator: sde of bridge
- the "absorbing" propagator: sde for avalanche

contact between FP and sde approach

S. N. Majumdar and H. Orland, J. Stat. Mech. (2015).
J. Pitman and M. Yor, A Guide to Brownian Motion and Related Stochastic Processes, (2018).
A. Mazzolo, J. Stat. Mech.(2017).

ABBM EFFECTIVE SDE FOR BRIDGE/EXCURSION

Doob's transform $$\frac{dv}{dt} = k\,(c-v) + \sigma^2\, v \lim_{\epsilon=0} \partial_v \log P(v,t \to \epsilon, T) + \sigma\sqrt{v}\xi(t)$$

Bridge: "free" propagator

$$\frac{dv}{dt} = k\,c - v\,k\,\coth\left[\frac{1}{2}k\,(T-t)\right] + \sigma\sqrt{v}\xi(t)$$

Avalanche: "absorbing" propagator

$$\frac{dv}{dt} = \sigma^2 - k\,c - v\,k\,\coth\left[\frac{1}{2}k\,(T-t)\right] + \sigma\sqrt{v}\xi(t)$$

Averageing sde equations

$$\langle\sqrt{v}\,\xi(t)\rangle = 0$$ (Ito scheme)

bridge:

$$\frac{d\langle v(t)\rangle}{dt} = k\,c - \langle v(t)\rangle\,k\,\coth\left[\frac{1}{2}k\,(T-t)\right]$$

avalanche:

$$\frac{d\langle v(t)\rangle}{dt} = \sigma^2 - k\,c - \langle v(t)\rangle\,k\,\coth\left[\frac{1}{2}k\,(T-t)\right]$$

Linear differential equations for average shapes!

ABBM AVERAGE BRIDGE AND AVALANCHE

Bridge:

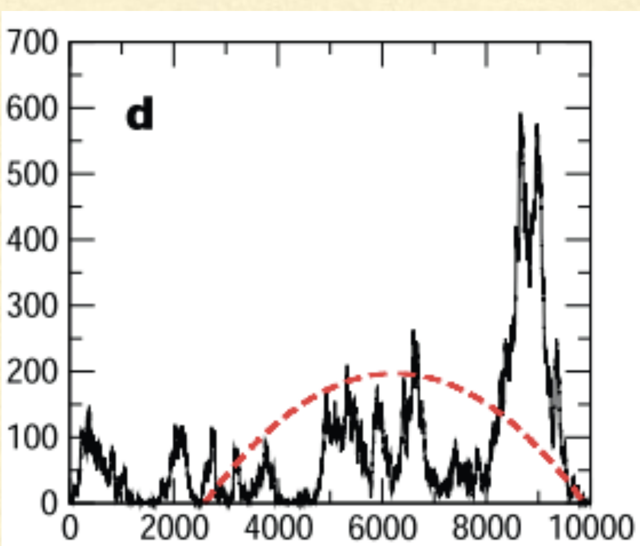
$$\langle v(t)\rangle_T = c\,\frac{\left(e^{-k\,t}-1\right)\left(e^{k\,(T-t)}-1\right)}{1-e^{-k\,T}}$$

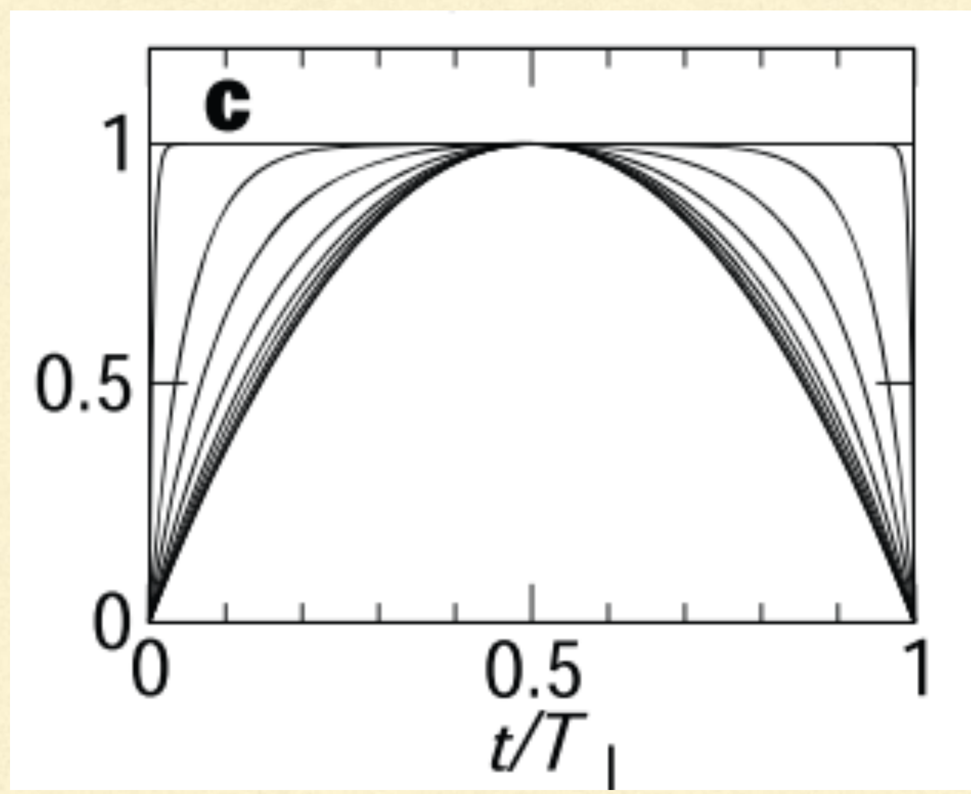
Avalanche

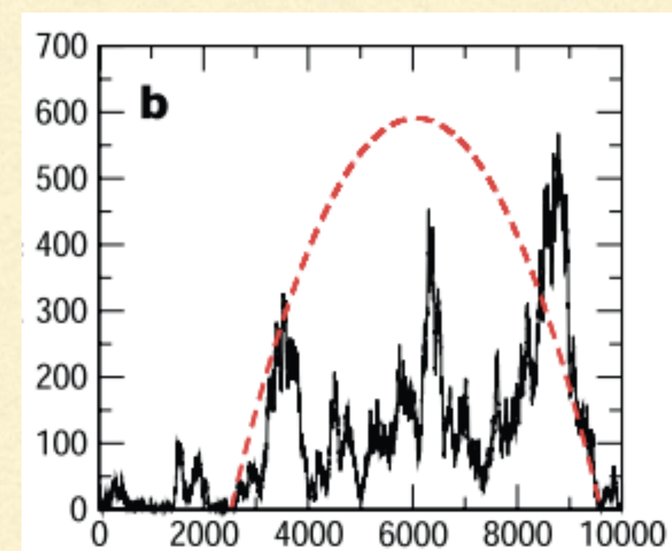
$$\langle v(t)\rangle_T = \left(\frac{\sigma^2}{k}-c\right)\frac{\left(e^{-k\,t}-1\right)\left(e^{k\,(T-t)}-1\right)}{1-e^{-k\,T}}$$

Same normalised shape!

$$\langle v(t)\rangle/\langle v(T/2)\rangle$$







symmetric average shapes: **parabolic** for $kT<<1$; flat for $kT>>1$

A. Baldassarri, J. Stat. Mech., 083211 (2021).

BRIDGE/EXCURSION DISTRIBUTIONS

Both gamma distribution, but with different *shape* parameter

$$\mu \equiv \frac{2kc}{\sigma^2} - 1$$

Bridge

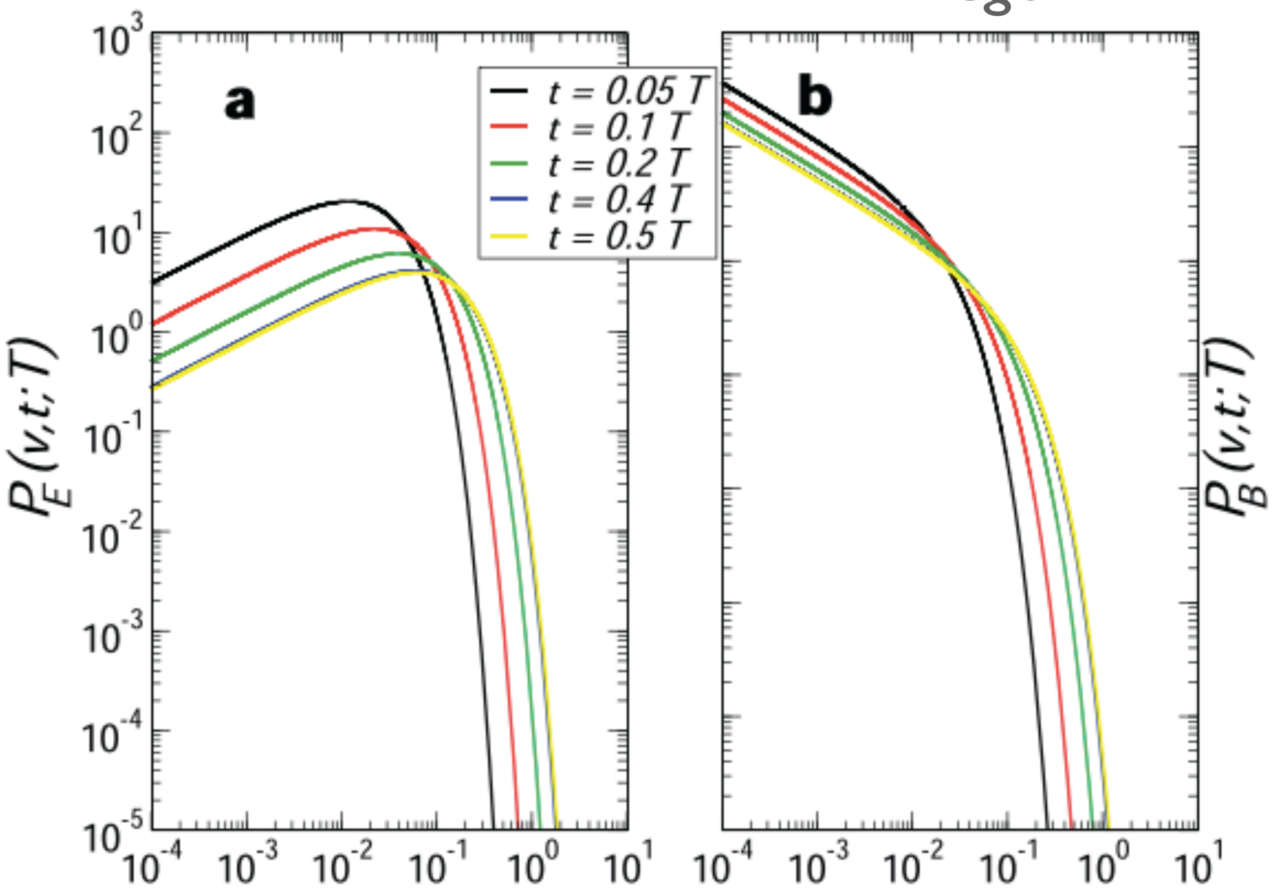
$$P_B(v,t;T) = \frac{\omega(t,T)^{1+\mu}}{\Gamma(1+\mu)} \exp\left(-\omega(t,T)\, v\right)\, v^{\mu}$$

Avalanche

$$P_E(v,t;T) = \frac{\omega(t,T)^{1-\mu}}{\Gamma(1-\mu)} \exp\left(-\omega(t,T)\, v\right)\, v^{-\mu}$$

varying *t*:

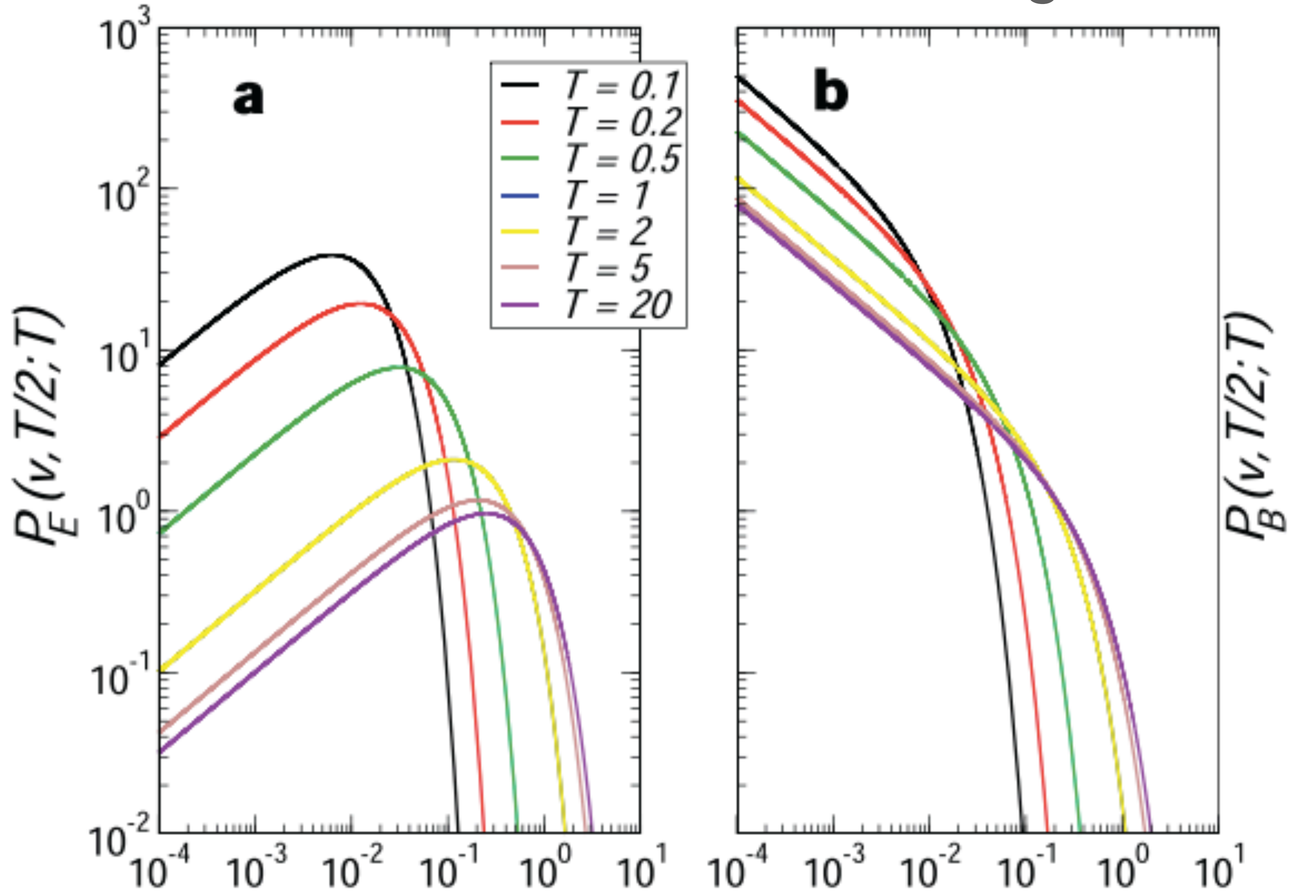
Avalanche *Bridge*



$\mu = -0.5,\ k = \sigma = T = 1$

varying *T*:

Avalanche *Bridge*



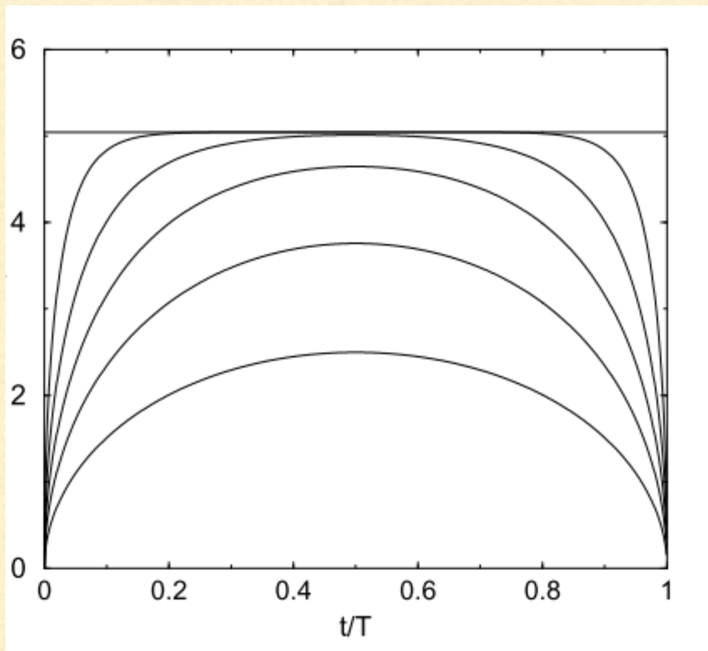
$\mu = -0.5,\ k = \sigma = 1,\ t = T/2$

SIMPLEST STOCHASTIC PROCESS

Dumped random walk (Ornstein-Uhlenbeck process)

$$\frac{dv(t)}{dt} = -k\,v + \xi(t) \qquad \overline{\xi(t)} = 0 \qquad \overline{\xi(t)\xi(s)} = \delta(t-s)$$

A. Baldassarri, F. Colaiori, and C. Castellano, PRL (2003).
F. Colaiori, A. Baldassarri, and C. Castellano, PRE (2004).



Exact formula

$$\langle v(t)\rangle_T = \sqrt{\frac{4}{\pi k}}\sqrt{\frac{\left(1-e^{-2kt}\right)\left(1-e^{-2k(T-t)}\right)}{1-e^{-2kT}}}$$

Limit cases:

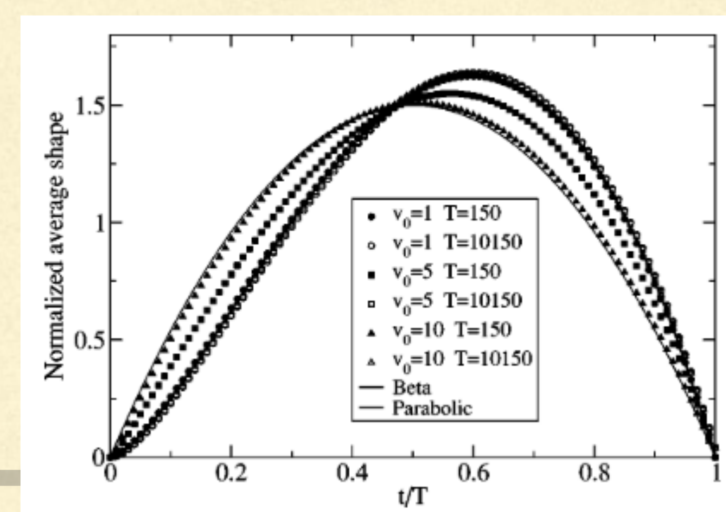
$$\langle v(t)\rangle_T \approx \begin{cases} \sqrt{\frac{8}{\pi}}\sqrt{\frac{t(T-t)}{T}} & T \ll 1/k \\ \sqrt{\frac{4}{k\pi}} & T \gg 1/k \end{cases}$$

RW: semicircle (Gaussian or Levy increments)

Uncorrelated: flat

"Random accelerated particle":

$$\frac{d^2v(t)}{dt^2} = \xi(t)$$



Different values of $\partial_t v(0)$

GENERALIZED BESSEL PROCESSES

$\vec{Y} = (Y_1, ..., Y_\delta)$ **δ**: space dimension

$$\frac{d\vec{Y}}{dt} = -\frac{\vec{Y}}{2} + \sqrt{2}\,\vec{\xi}(t)$$ **δ**-dimensional Ornstein-Uhlenbeck process

$v(t) = |\vec{Y}(t)|^2$ $\qquad \frac{dv}{dt} = \left(\frac{\delta}{2} - v\right) + \sqrt{2\,v}\,\xi(t)$ Generalized Squared Bessel Process

equivalent to $\frac{dv}{dt} = (c - v) + \sqrt{2v}\xi(t)$ ABBM/CIR process $c = \frac{\delta}{2}$

Return to the origin?

$$v = 0 \iff \vec{Y} = (Y_1 = 0, ..., Y_\delta = 0) = \vec{0}$$

ABBM/CIR	*drive*		*dimension*	**Random Walk**
stick-slip	$c < 1$	$\iff$	$\delta < 2$	recurrent
steady-sliding	$c > 1$	$\iff$	$\delta > 2$	transient

ASYMMETRIC AVALANCHE SHAPES

INERTIAL ABBM

S. Zapperi, C. Castellano, F. Colaiori, and G. Durin, Nat. Phys.(2005).
G. Durin, F. Colaiori, C. Castellano, and S. Zapperi, J. Magn. Magn. Mater.(2007)

$$\int_0^t f(t-s)v(s) = k(ct-x) + F(x)$$

Generalised damping term for non-instantaneous response due to eddy currents

if $f(t) \approx \frac{\Gamma}{\tau_0} e^{-\frac{t}{\tau_0}}$

and the avalanche is longer than τ_0

then $\frac{\Gamma}{\tau_0} \int_0^t e^{-s/\tau_0} v(t-s) \approx \Gamma v(t) - \Gamma \tau_0 \frac{dv}{dt}$ that gives:

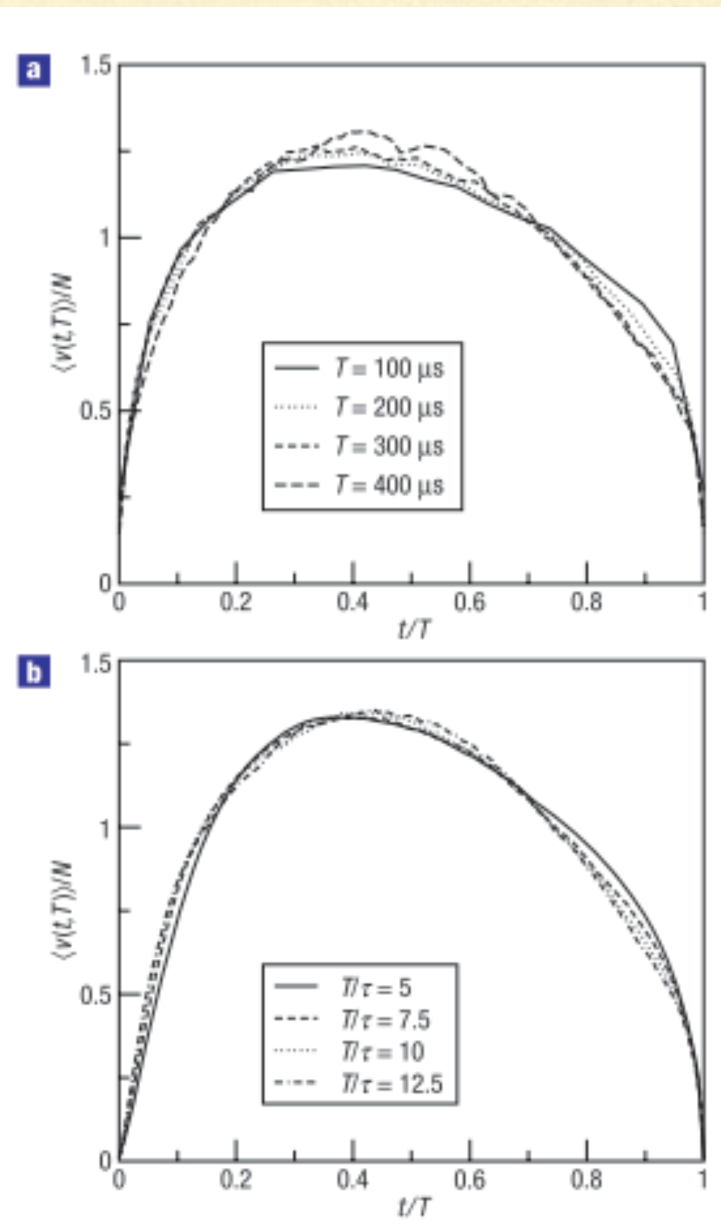
$$\Gamma v + M \frac{dv}{dt} = k(ct-x) + F(x)$$

inertial term

but

$$M = -\frac{64 I^2 \sigma b^2}{a \pi^4} \Sigma_2(b/a) < 0$$

negative mass

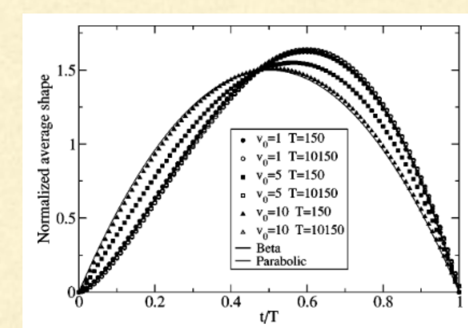


Experiments

Model

opposite skewness to the case of positive mass

$$\frac{d^2 x}{dt^2} = \xi(t)$$



see also: P. Le Doussal, A. Petković, and K. J. Wiese, Phys. Rev. E (2012)
A. Dobrinevski, P. Le Doussal, and K. J. Wiese, Phys. Rev. E(2013)

AVERAGE SHAPE UNIVERSALITY CLASSES

L. Laurson, X. Illa, S. Santucci, K. Tore Tallakstad, K. J. Måløy, and M. J. Alava, Class., Nat. Commun. (2013).

Simulation of a model of 1d-elastic string in a 2d random medium

Average shape proposal

$$\langle S(T)\rangle \propto T^{\gamma} \qquad \langle v(t)\rangle_T \propto T^{\gamma-1}\left[\frac{t}{T}\left(1-\frac{t}{T}\right)\right]^{\gamma-1}\left[1-a\left(\frac{t}{T}-\frac{1}{2}\right)\right]$$

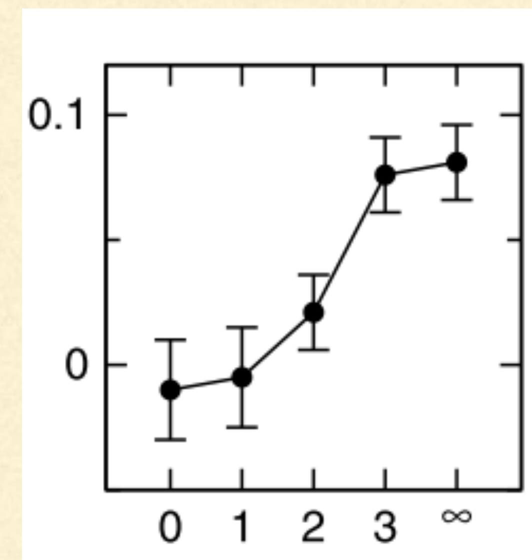
Universality classes:

scaling exponent

$$\begin{aligned} \gamma &= 2.0 \pm 0.01 && \text{for} \quad \alpha \le 1 \\ \gamma &= 1.79 \pm 0.01 && \text{for} \quad \alpha = 2 \\ \gamma &= 1.56 \pm 0.01 && \text{for} \quad \alpha \ge 3 \end{aligned}$$

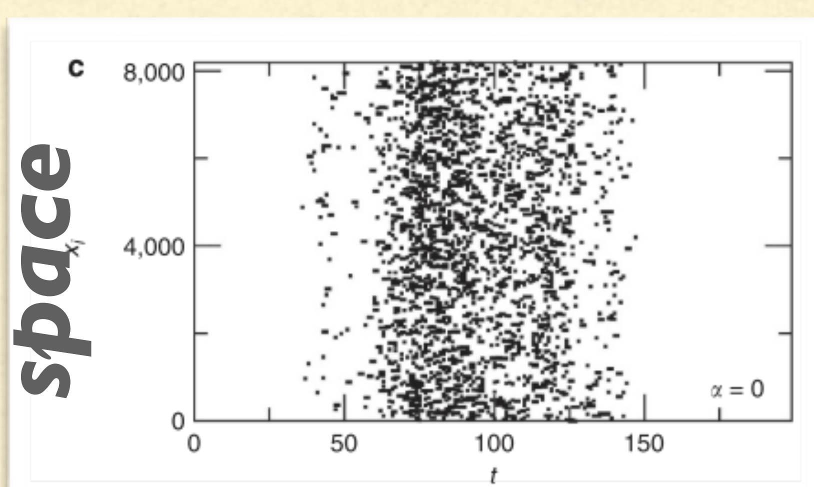
asymmetry parameter

a



α

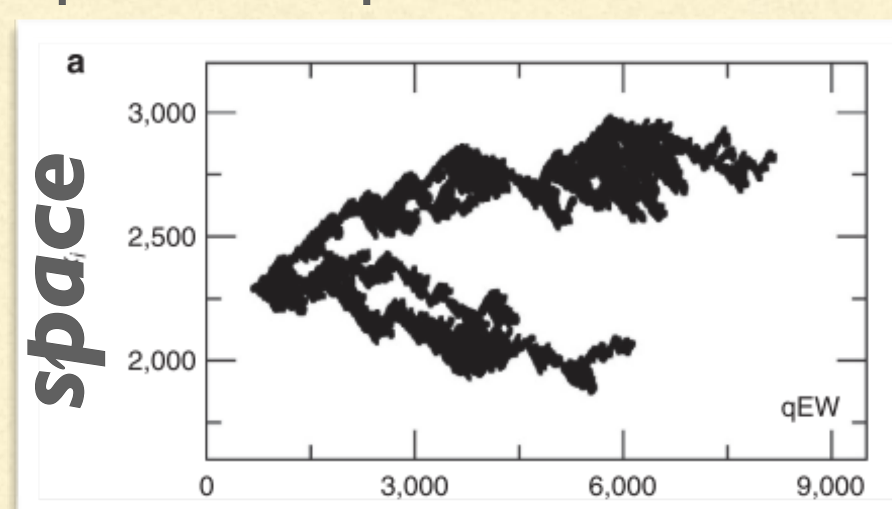
Asymmetry related to *time irreversibility* of spatio-temporal structure of avalanches



$\alpha = 0$

$a \approx 0$

time



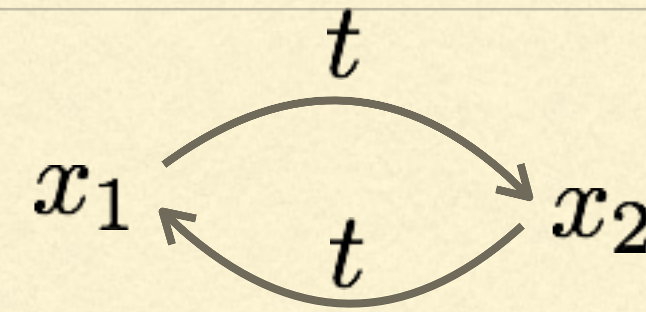
$\alpha = 3$

$a \approx 0.1$

time

DETAILED BALANCE

DETAILED BALANCE



$$P_s(x_1)P(x_1 \xrightarrow{t} x_2) = P_s(x_2)P(x_2 \xrightarrow{t} x_1)$$

general bridge

$$B(x,t|x_0 \xrightarrow{\tau} x_f) = \frac{P(x_0 \xrightarrow{t} x)P(x \xrightarrow{\tau - t} x_f)}{P(x_0 \xrightarrow{\tau} x_f)}$$

use d.b.

$$B(x,t|x_0 \xrightarrow{\tau} x_f) = \frac{P(x \xrightarrow{t} x_0)\frac{P_s(x)}{P_s(x_0)}P(x_f \xrightarrow{\tau - t} x)\frac{P_s(x_f)}{P_s(x)}}{P(x_f \xrightarrow{\tau} x_0)\frac{P_s(x_f)}{P_s(x_0)}}$$

simplify

$$B(x,t|x_0 \xrightarrow{\tau} x_f) = \frac{P(x \xrightarrow{t} x_0)P(x_f \xrightarrow{\tau - t} x)}{P(x_f \xrightarrow{\tau} x_0)} = B(x,\tau - t|x_f \xrightarrow{\tau} x_0)$$

bridge $x_0 = x_f = 0$ $\quad B(x,t|0 \xrightarrow{\tau} 0) = B(x,\tau - t|0 \xrightarrow{\tau} 0)$

moments $\quad \langle x(t)^2\rangle_\tau = \langle x(\tau - t)^2\rangle_\tau$

d.b. implies symmetry of bridge average shape

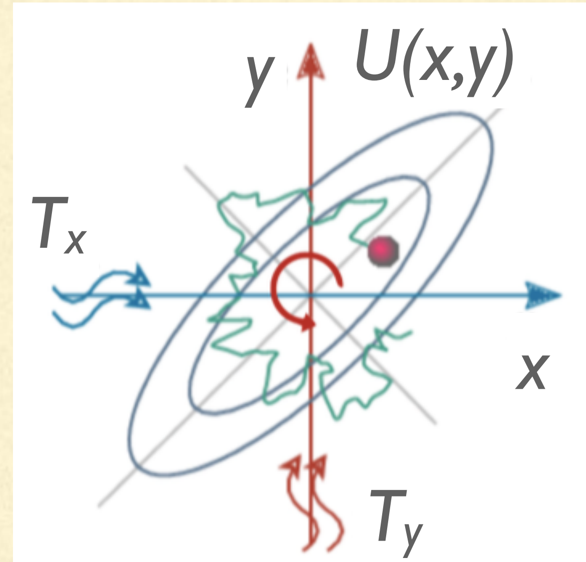
BROWNIAN GYRATOR

R. Filliger and P. Reimann, PRL (2007).

$$\frac{dx}{dt} = -k\,x - u\,y + \sqrt{T_x}\,\xi_1(t)$$

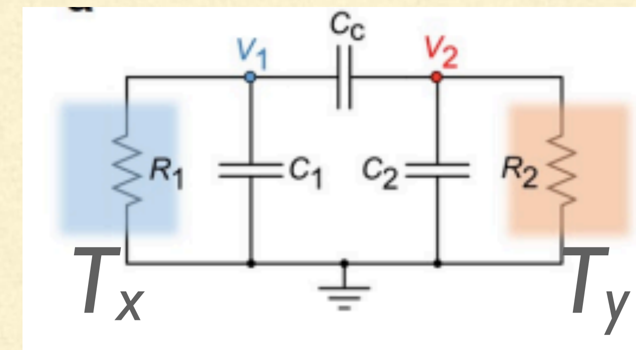
$$\frac{dy}{dt} = -k\,y - u\,x + \sqrt{T_y}\,\xi_2(t)$$

$\langle \xi_i(t) \rangle = 0$ and $\langle \xi_i(t)\xi_j(s) \rangle = \delta_{ij}\,\delta(t-s)$



$$U(x,y) = \frac{1}{2}k\left(x^2+y^2\right) + u\,x\,y$$

experimental realization



K. H. Chiang, C. L. Lee, P. Y. Lai, and Y. F. Chen, PRE (2017).

FP equation

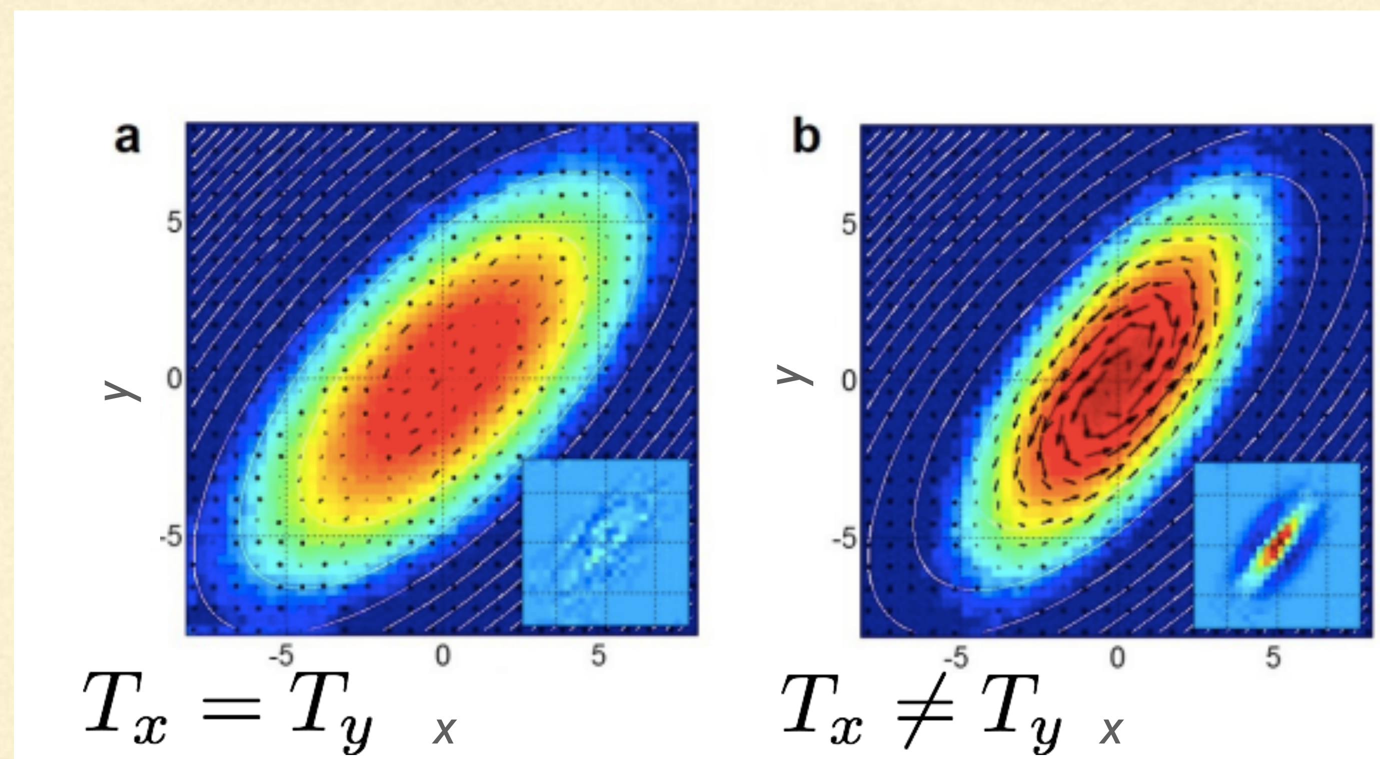
$$\partial_t P(x,y,t) = \vec{\nabla}\cdot\vec{J}(x,y,t)$$

Stationary state

$$\lim_{t\to\infty} P(x,y,t) = P_s(x,y)$$

$$\lim_{t\to\infty} \vec{J}(x,y,t) = \vec{J}_s(x,y)$$

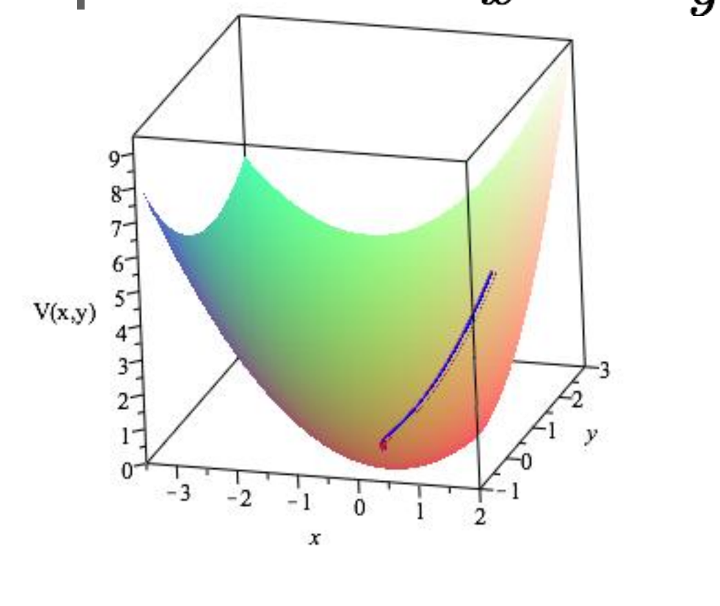
$$\vec{\nabla}\cdot\vec{J}_s(x,y) = 0$$



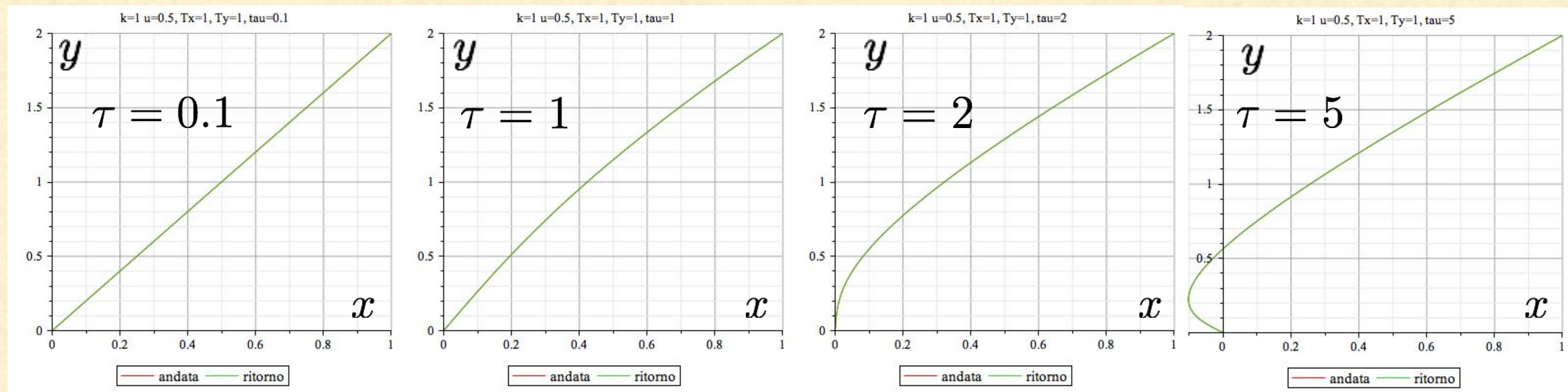
Equilibrium $J_s(x,y) = 0$ NESS $J_s(x,y) \neq 0$

AVERAGE BRIDGE FOR BROWNIAN GYRATOR

Equilibrium $T_x = T_y$



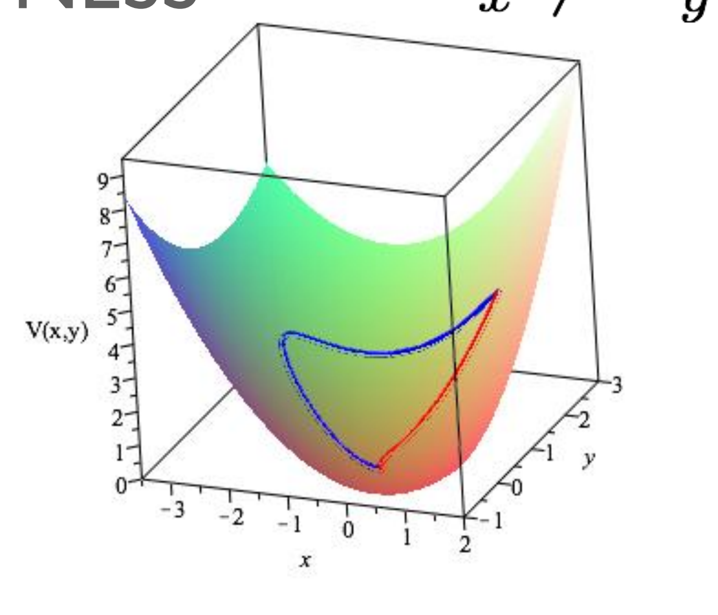
Average bridge $(0,0) \to (1,2)$ in a time τ

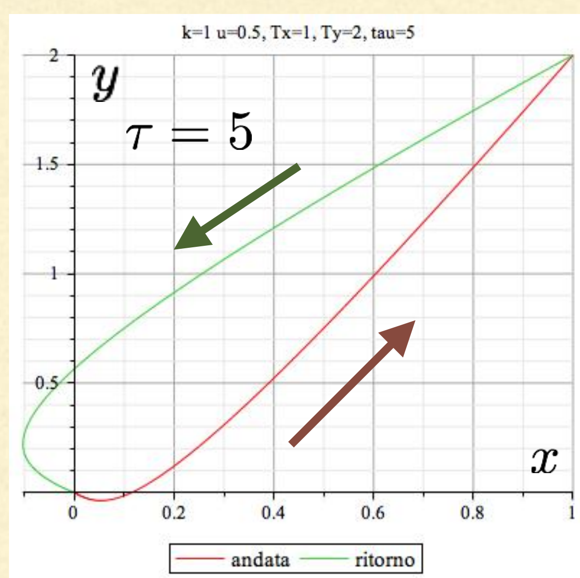


same average path in direct and reverse bridge

$$(0,0) \xrightarrow{\tau} (1,2) \text{ vs } (1,2) \xrightarrow{\tau} (0,0)$$

NESS $T_x \neq T_y$





No detailed balance

different average path in direct and reverse bridge

D. Lucente, A. Baldassarri, A. Puglisi, A. Vulpiani, M. Viale, arXiv:2205.08961 (2022)

ASYMMETRIC BRIDGE FOR SINGLE COMPONENT

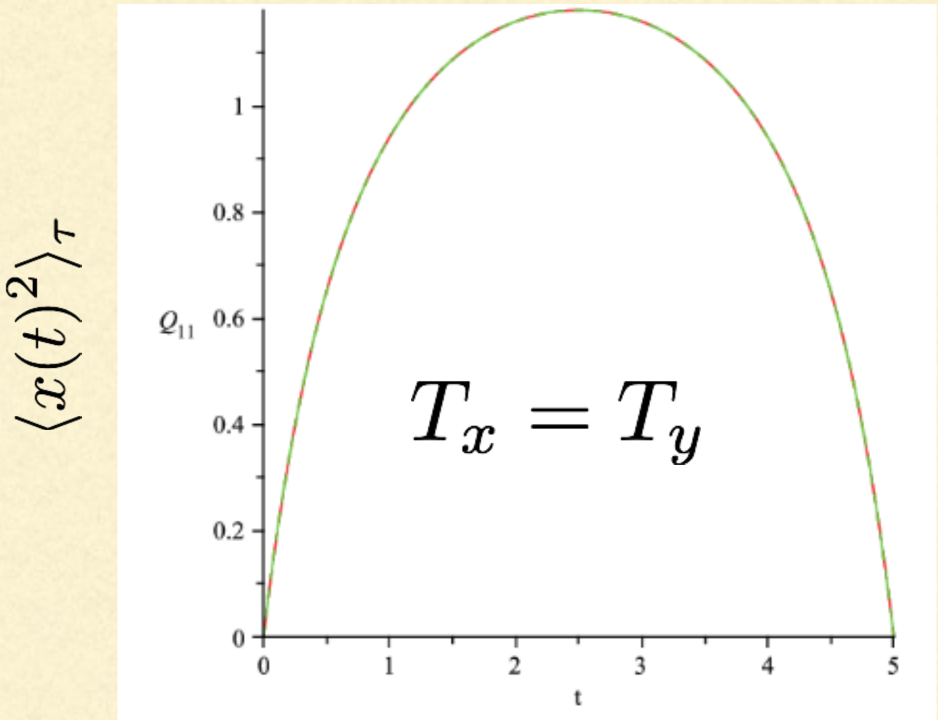
$B(x, y, t \,|\, x_1, y_1 \xrightarrow{\tau} x_2, y_2)$ Full bridge distribution

Bridge in x,y plane $(0,0) \xrightarrow{\tau} (0,0)$

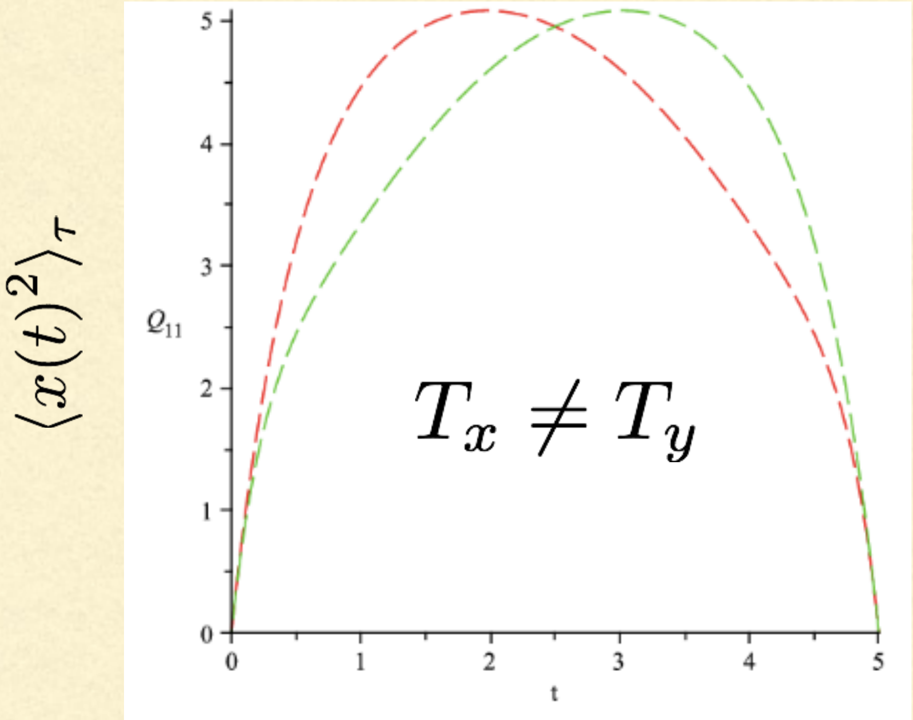
single component average shape

$$\langle x^2(t) \rangle_T = \int dy \int dx \;\; x^2 \; B(x, y, t \,|\, x_1, y_1 \xrightarrow{\tau} x_2, y_2)$$

(first moment null)



symmetric average shape



asymmetric average shape

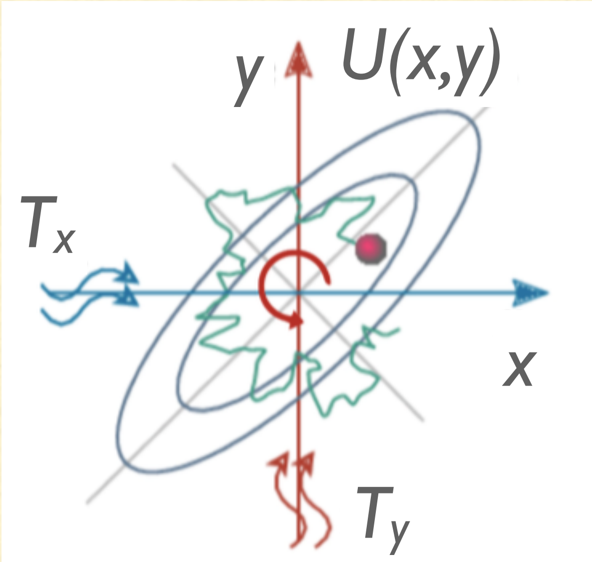
NON MARKOV PROCESS: INCOMPLETE KNOWLEDGE

INCOMPLETE KNOWLEDGE

Brownian gyrator

$$\frac{dx}{dt} = -k\,x - \alpha\,y + \sqrt{T_x}\,\xi_1(t)$$
$$\frac{dy}{dt} = -k\,y - \alpha\,x + \sqrt{T_y}\,\xi_1(t)$$

(x, y) is Markov



What if we don't have any access to *y(t)*?

Look at *x(t)* only:

"effective propagator"? *how to average over hidden variables?*

$$P_s(x_1 \xrightarrow{t} x) = \int dy_1 P_s(x_1, y_1) \int dy P(x_1, y_1 \xrightarrow{t} x, y)$$

stationary initial condition — *marginal final condition*

(x) is not Markov *Hidden variable y contribute to memory of x(t)*

$$P_s(x_0, t_0;\, x_1, t_1;\, \dots x_n, t_n) \neq P_s(x_0) P_s(x_0, \xrightarrow{t_1 - t_0} x_1) \dots P_s(x_{n-1} \xrightarrow{t_n - t_{n-1}} x_n)$$

SHAPE WITH INCOMPLETE KNOWLEDGE

bridge distribution

no more: $B_s(x,t|\,x_1 \xrightarrow{\tau} x_2) \neq \dfrac{P_s(x_1 \xrightarrow{t} x)P_s(x \xrightarrow{\tau-t} x_2)}{P_s(x_1 \xrightarrow{\tau} x_2)}$

but

$$B_s(x,t|\,x_1 \xrightarrow{\tau} x_2) = \frac{\int dy_1\, dy_2\, dy P_s(x_1,y_1)P(x,y \xrightarrow{t} x,y)P(x,y \xrightarrow{\tau-t} x_2,y_2)}{\int dy_1 dy_2 P_s(x_1,y_1)P(x_1,y_1 \xrightarrow{\tau} x_2,y_2)}$$

complicated expression. However, still:

detailed balance of complete system

$$P_s(x_1,y_1)P(x_1,y_1 \xrightarrow{t} x_2,y_2) = P_s(x_2,y_2)P(x_2,y_2 \xrightarrow{t} x_1,y_1)$$

implies $\Downarrow$

symmetry of the single component average shape

$$B_s(x,t\,|\,x_1 \xrightarrow{\tau} x_2) = B_s(x,\tau-t\,|x_2 \xrightarrow{\tau} x_1)$$

$$\langle x(t)^2\rangle_\tau = \langle x(\tau-t)^2\rangle_\tau$$

HOPES…

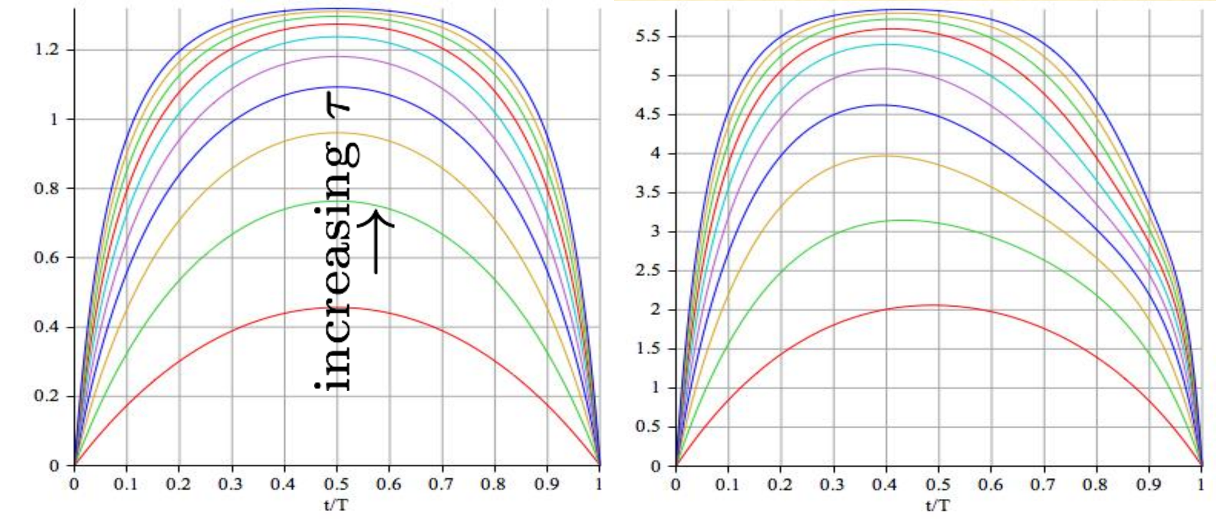
- detailed balance implies symmetric shape
- broken d.b. **does not imply** asymmetric shape, however:
- B.G. complete system does show asymmetric shape when d.b. is broken
- single component shape is obtained with an asymmetric average procedure over the hidden variables (stationary initial condition, marginal final condition)
- *how can an asymmetric complicated average procedure restore symmetry of the single component average shape?*

PITFALL FOR LINEAR SYSTEMS

Complete knowledge

$$(0,0) \xrightarrow{\tau} (0,0)$$

$T_x = T_y$ $\qquad$ $T_x \neq T_y$

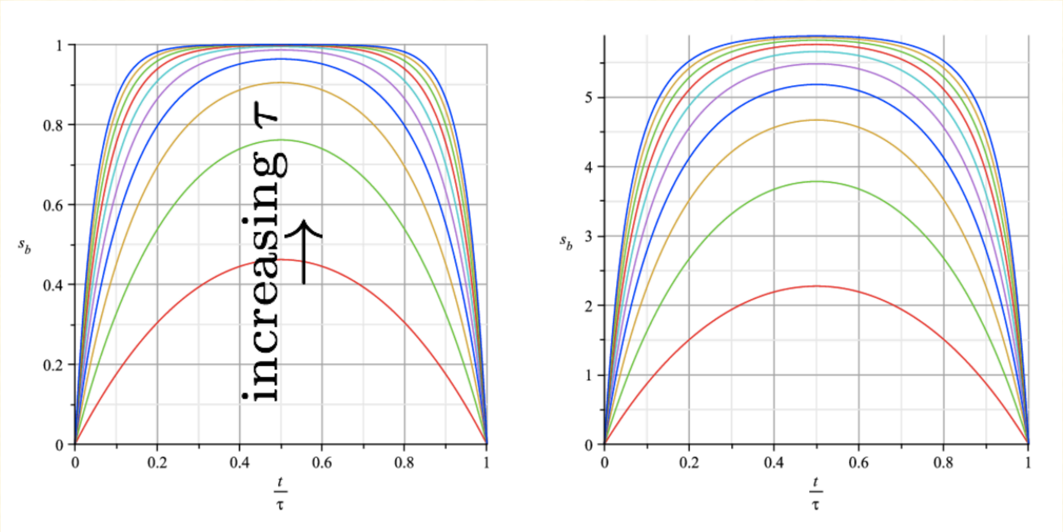


asymmetry signals non equilibrium dynamics

Incomplete knowledge

$$0 \xrightarrow{\tau} 0$$

$T_x = T_y$ $\qquad$ $T_x \neq T_y$



shape always symmetric!
we can not understand if the system is in equilibrium or not

CONCLUSION AND OUTLOOKS

The shape of a fluctuation (avalanche or bridge) may contains interesting information, not yet exploited:

- Avalanche shape is more suitable for universality features (scaling exponents)
- Bridge shape may be statistically more significant and analytically approachable
- Can bridge statistics reveal correlation btw avalanches? (e.g. Omori's law?)

Beyond universality, one may hope to extract from fluctuation (avalanche or bridge) shapes some "stochastic thermodynamic" information on out-of equilibrium systems:

- Linear systems can be easy to solve, but trivial or misleading
- Since broken d.b. allows for asymmetric shapes, can we relate some asymmetric measure to entropy production rate of the (full) system?

[1] A. Baldassarri, J. Stat. Mech., 083211 (2021).
[2] A. Baldassarri, M.A. Annunziata, A. Gnoli, G. Pontuale, A. Petri, Sci Rep 9, 16962 (2019).
[3] D. Lucente, A. Baldassarri, A. Puglisi, A. Vulpiani, M. Viale, arXiv:2205.08961 (2022).

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